THE ARCHITECTURE OF MATHEMATICS*

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1. Mathematic or mathematics? To present a view of the entire field of mathematical science as it exists,—this is an enterprise which presents, at first sight, almost insurmountable difficulties, on account of the extent and the varied character of the subject. As is the case in all other sciences, the number of mathematicians and the number of works devoted to mathematics have greatly increased since the end of the 19th century. The memoirs in pure mathematics published in the world during a normal year cover several thousands of pages. Of course, not all of this material is of equal value; but, after full allowance has been made for the unavoidable tares, it remains true nevertheless that mathematical science is enriched each year by a mass of new results, that it spreads and branches out steadily into theories, which are subjected to modifications based on new foundations, compared and combined with one another. No mathematician, even were he to devote all his time to the task, would be able to follow all the details of this development. Many mathematicians take up quarters in a corner of the domain of mathematics, which they do not intend to leave; not only do they ignore almost completely what does not concern their special field, but they are unable to understand the language and the terminology used by colleagues who are working in a corner remote from their own. Even among those who have the widest training, there are none who do not feel lost in certain regions of the immense world of mathematics; those who, like Poincaré or Hilbert, put the seal of their genius on almost every domain, constitute a very great exception even among the men of greatest accomplishment.

It must therefore be out of the question to give to the uninitiated an exact picture of that which the mathematicians themselves can not conceive in its totality. Nevertheless it is legitimate to ask whether this exuberant proliferation makes for the development of a strongly constructed organism, acquiring ever greater cohesion and unity with its new growths, or whether it is the external manifestation of a tendency towards a progressive splintering, inherent in the very nature of mathematics, whether the domain of mathematics is not becoming a tower of Babel, in which autonomous disciplines are being more and more widely separated from one another, not only in their aims, but also in their methods and even in their language. In other words, do we have today a mathematic or do we have several mathematics?

Although this question is perhaps of greater urgency now than ever before, it is by no means a new one; it has been asked almost from the very beginning of mathematical science. Indeed, quite apart from applied mathematics, there has

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always existed a dualism between the origins of geometry and of arithmetic (certainly in their elementary aspects), since the latter was at the start a science of discrete magnitude, while the former has always been a science of continuous extent; these two aspects have brought about two points of view which have been in opposition to each other since the discovery of irrationals. Indeed, it is exactly this discovery which defeated the first attempt to unify the science, *viz.*, the arithmetization of the Pythagoreans ("everything is number").

It would carry us too far if we were to attempt to follow the vicissitudes of the unitary conception of mathematics from the period of Pythagoras to the present time. Moreover this task would suit a philosopher better than a mathematician; for it is a common characteristic of the various attempts to integrate the whole of mathematics into a coherent whole—whether we think of Plato, of Descartes or of Leibnitz, of arithmetization, or of the logistics of the 19th century—that they have all been made in connection with a philosophical system, more or less wide in scope; always starting from *a priori* views concerning the relations of mathematics with the twofold universe of the external world and the world of thought. We can do no better on this point than to refer the reader to the historical and critical study of L. Brunschvicg [1]. Our task is a more modest and a less extensive one; we shall not undertake to examine the relations of mathematics to reality or to the great categories of thought; we intend to remain within the field of mathematics and we shall look for an answer to the question which we have raised, by analyzing the procedures of mathematics themselves.

2. Logical formalism and axiomatic method. After the more or less evident bankruptcy of the different systems, to which we have referred above, it looked, at the beginning of the present century as if the attempt had just about been abandoned to conceive of mathematics as a science characterized by a definitely specified purpose and method; instead there was a tendency to look upon mathematics as "a collection of disciplines based on particular, exactly specified concepts," interrelated by "a thousand roads of communication," allowing the methods of any one of these disciplines to fertilize one or more of the others [1, page 447]. Today, we believe however that the internal evolution of mathematical science has, in spite of appearance, brought about a closer unity among its different parts, so as to create something like a central nucleus that is more coherent than it has ever been. The essential aspect of this evolution has been the systematic study of the relations existing between different mathematical theories, and which has led to what is generally known as the "axiomatic method."

The words "formalism" and "formalistic method" are also often used; but it is important to be on one's guard from the start against the confusion which may be caused by the use of these ill-defined words, and which is but too frequently made use of by the opponents of the axiomatic method. Everyone knows that superficially mathematics appears as this "long chain of reasons" of

which Descartes spoke; every mathematical theory is a concatenation of propositions, each one derived from the preceding ones in conformity with the rules of a logical system, which is essentially the one codified, since the time of Aristotle, under the name of "formal logic," conveniently adapted to the particular aims of the mathematician. It is therefore a meaningless truism to say that this "deductive reasoning" is a unifying principle for mathematics. So superficial a remark can certainly not account for the evident complexity of different mathematical theories, not any more than one could, for example, unite physics and biology into a single science on the ground that both use the experimental method. The method of reasoning by means of chains of syllogisms is nothing but a transforming mechanism, applicable just as well to one set of premises as to another; it could not serve therefore to characterize these premises. In other words, it is the external form which the mathematician gives to his thought, the vehicle which makes it accessible to others,* in short, the language suited to mathematics; this is all, no further significance should be attached to it. To lay down the rules of this language, to set up its vocabulary and to clarify its syntax, all that is indeed extremely useful; indeed this constitutes one aspect of the axiomatic method, the one that can properly be called logical formalism (or "logistics" as it is sometimes called). But we emphasize that it is but one aspect of this method, indeed the least interesting one.

What the axiomatic method sets as its essential aim, is exactly that which logical formalism by itself can not supply, namely the profound intelligibility of mathematics. Just as the experimental method starts from the *a priori* belief in the permanence of natural laws, so the axiomatic method has its cornerstone in the conviction that, not only is mathematics not a randomly developing concatenation of syllogisms, but neither is it a collection of more or less "astute" tricks, arrived at by lucky combinations, in which purely technical cleverness wins the day. Where the superficial observer sees only two, or several, quite distinct theories, lending one another "unexpected support" [1, page 446] through the intervention of a mathematician of genius, the axiomatic method teaches us to look for the deep-lying reasons for such a discovery, to find the common ideas of these theories, buried under the accumulation of details properly belonging to each of them, to bring these ideas forward and to put them in their proper light.

3. The notion of structure. In what form can this be done? It is here that the axiomatic method comes closest to the experimental method. Like the latter drawing its strength from the source of Cartesianism, it will "divide the difficulties in order to overcome them better." It will try, in the demonstrations of a theory, to separate out the principal mainsprings of its arguments; then, taking each of these separately and formulating it in abstract form, it will develop

* Indeed every mathematician knows that a proof has not really been "understood" if one has done nothing more than verifying step by step the correctness of the deductions of which it is composed, and has not tried to gain a clear insight into the ideas which have led to the construction of this particular chain of deductions in preference to every other one.

the consequences which follow from it alone. Returning after that to the theory under consideration, it will recombine the component elements, which had previously been separated out, and it will inquire how these different components influence one another. There is indeed nothing new in this classical going to-and-fro between analysis and synthesis; the originality of the method lies entirely in the way in which it is applied.

In order to illustrate the procedure which we have just sketched, by an example, we shall take one of the oldest (and also one of the simplest) of axiomatic theories, *viz.* that of the "abstract groups." Let us consider for example, the three following operations: 1. the addition of real numbers, their sum (positive negative or zero) being defined in the usual manner; 2. the multiplication of integers "modulo a prime number p," (where the elements under consideration are the whole numbers $1, 2, \cdots, p-1$) and the "product" of two of these numbers is, by agreement, defined as the remainder of the division of their usual product by p; 3. the "composition" of displacements in three-dimensional Euclidean space, the "resultant" (or "product") of two displacements S, T (taken in this order) being defined as the displacement obtained by carrying out first the displacement T and then the displacement S. In each of these three theories, one makes correspond, by means of a procedure defined for each theory, to two elements x, y (taken in that order) of the set under consideration (in the first case the set of real numbers, in the second the set of numbers $1, 2, \cdots, p-1$, in the third the set of all displacements) a well-determined third element; we shall agree to designate this third element in all three cases by $x\tau y$ (this will be the sum of x and y if x and y are real numbers, their product "modulo p" if they are integers $\leqq p-1$, their resultant if they are displacements). If we now examine the various properties of this "operation" in each of the three theories, we discover a remarkable parallelism; but, in each of the separate theories, the properties are interconnected, and an analysis of their logical connections leads us to select a small number of them which are independent (*i.e.*, none of them is a logical consequence of all the others). For example,* one can take the three following, which we shall express by means of our symbolic notation, common to the three theories, but which it would be very easy to translate into the particular language of each of them:

(a) For all elements x, y, z, one has $x\tau(y\tau z)=(x\tau y)\tau z$ ("associativity" of the operation $x\tau y$);

(b) There exists an element e, such that for every element x, one has $e\tau x=x\tau e=x$ (for the addition of real numbers, it is the number 0; for multiplication "modulo p," it is the number 1; for the composition of displacements, it is the "identical" displacement, which leaves every point of space fixed);

(c) Corresponding to every element x, there exists an element x' such that $x\tau x'=x'\tau x=e$ (for the addition of real numbers x' is the number $-x$; for the

* There is nothing absolute in this choice; several systems of axioms are known which are "equivalent" to the one which we are stating explicitly, the axioms of each of these systems being logical consequences of the axioms of any other one.

composition of displacements, x' is the "inverse" displacement of x, *i.e.* the displacement which replaces each point that had been displaced by x to its original position; for multiplication "modulo p," the existence of x' follows from a very simple arithmetic argument.*

It follows then that the properties which can be expressed in the same way in the three theories, by means of the common notation, are consequences of the three preceding ones. Let us try to show, for example that from $x\tau y = x\tau z$ follows $y = z$; one could do this in each of the theories by a reasoning peculiar to it. But, we can proceed as follows by a method that is applicable in all cases: from the relation $x\tau y = x\tau z$ we derive (x' having the meaning which was defined above) $x'\tau(x\tau y) = x'\tau(x\tau z)$; thence by applying (a), $(x'\tau x)\tau y = (x'\tau x)\tau z$; by means of (c), this relation takes the form $e\tau y = e\tau z$, and finally, by applying (b), $y = z$, which was to be proved. In this reasoning the nature of the elements x, y, z under consideration has been left completely out of account; we have not been concerned to know whether they are real numbers, or integers $\leqq p-1$, or displacements; the only premise that was of importance was that the operation $x\tau y$ on these elements has the properties (a), (b), and (c). Even if it were only to avoid irksome repetitions, it is readily seen that it would be convenient to develop once and for all the logical consequences of the three properties (a), (b), (c) only. For linguistic convenience, it is of course desirable to adopt a common terminology for the three sets. One says that a set in which an operation $x\tau y$ has been defined which has the three properties (a), (b), (c) is provided with a group structure (or, briefly, that it is a group); the properties (a), (b), (c) are called the axioms of** the group structures, and the development of their consequences constitutes setting up the axiomatic theory of groups.

It can now be made clear what is to be understood, in general, by a mathematical structure. The common character of the different concepts designated by this generic name, is that they can be applied to sets of elements whose nature† has not been specified; to define a structure, one takes as given one or

* We observe that the remainders left when the numbers x, x^2, $\cdots$, x^n, $\cdots$ are divided by p, can not all be distinct; by expressing the fact that two of these remainders are equal, one shows easily that a power x^m of x exists which has a remainder equal to 1; if now x' is the remainder of the division of x^{m-1} by p, we conclude that the product "modulo p" of x and x' is equal to 1.

** It goes without saying that there is no longer any connection between this interpretation of the word "axiom" and its traditional meaning of "evident truth."

† We take here a naive point of view and do not deal with the thorny questions, half philosophical, half mathematical, raised by the problem of the "nature" of the mathematical "beings" or "objects." Suffice it to say that the axiomatic studies of the nineteenth and twentieth centuries have gradually replaced the initial pluralism of the mental representation of these "beings"—thought of at first as ideal "abstractions" of sense experiences and retaining all their heterogeneity—by an unitary concept, gradually reducing all the mathematical notions, first to the concept of the natural number and then, in a second stage, to the notion of set. This latter concept, considered for a long time as "primitive" and "undefinable," has been the object of endless polemics, as a result of its extremely general character and on account of the very vague type of mental representation which it calls forth; the difficulties did not disappear until the notion of set itself disappeared (and with it all the metaphysical pseudo-problems concerning mathematical "beings" in the light of the recent work on logical formalism. From this new point of view, mathematical

several relations, into which these elements enter* (in the case of groups, this was the relation $z = x\tau y$ between three arbitrary elements); then one postulates that the given relation, or relations, satisfy certain conditions (which are explicitly stated and which are the axioms of the structure under consideration.)† To set up the axiomatic theory of a given structure, amounts to the deduction of the logical consequences of the axioms of the structure, excluding every other hypothesis on the elements under consideration (in particular, every hypotheses as to their own nature).

4. The great types of structures. The relations which form the starting point for the definition of a structure can be of very different characters. The one which occurs in the group structure is what one calls a "law of composition," *i.e.*, a relation between three elements which determines the third uniquely as a function of the first two. When the relations which enter the definition of a structure are "laws of composition," the corresponding structure is called an algebraic structure (for example, a field structure is defined by two laws of composition, with suitable axioms: the addition and multiplication of real numbers define a field structure on the set of these numbers).

Another important type is furnished by the structures defined by an order relation; this is a relation between two elements x, y which is expressed most frequently in the form "x is at most equal to y," and which we shall represent in general by $x\mathrm{R}y$. It is not at all supposed here that it determines one of the two elements x, y uniquely as a function of the other; the axioms to which it is subjected are the following: (a) for every x we have $x\mathrm{R}x$; (b) from the relations $x\mathrm{R}y$ and $y\mathrm{R}x$ follows $x = y$; (c) the relations $x\mathrm{R}y$ and $y\mathrm{R}z$ have as a consequence $x\mathrm{R}z$. An obvious example of a set with a structure of this kind is the set of integers (or that of real numbers), when the symbol R is replaced by the symbol $\leqq$. But it must be observed that we have not included among the axioms the following property, which seems to be inseparable from the popular notion of "order," "for every pair of elements x and y, either $x\mathrm{R}y$ or $y\mathrm{R}x$ holds." In other words, the case in which x and y are incomparable is not excluded. This may seem paradoxical at first sight, but it is easy to give examples of very important order structures, in which such a phenomenon appears. This is what happens when X and Y denote parts of the same set and the relation $X\mathrm{R}Y$ is interpreted to mean "X is contained in Y"; again when x and y are positive integers and $x\mathrm{R}y$ means

structures become, properly speaking, the only "objects" of mathematics. The reader will find fuller developments of this point in articles by J. Dieudonné [2] and H. Cartan [3].

* In effect, this definition of structures is not sufficiently general for the needs of mathematics; it is also necessary to consider the case in which the relations which define a structure hold not between elements of the set under consideration, but also between parts of this set and even, more generally, between elements of sets of still higher "degree" in the terminology of the "hierarchy of types." For further details on this point, see [4].

† Strictly speaking, one should, in the case of groups, count among the axioms, besides properties (a), (b), (c) stated above, the fact that the relation $z = x\tau y$ determines one and only one z when x and y are given; one usually considers this property as tacitly implied by the form in which the relation is written.

"x divides y"; also if $f(x)$ and $g(x)$ are real-valued functions defined on an interval $a \leqq x \leqq b$, while $f(x)\mathrm{R}g(x)$ is interpreted to mean "for every x, $f(x) \leqq g(x)$." These examples also give an indication of the great variety of domains in which order structures appear and thus point to the interest attached to their study.

We want to say a few words about a third large type of structures, *viz.* topological structures (or topologies): they furnish an abstract mathematical formulation of the intuitive concepts of neighborhood, limit and continuity, to which we are led by our idea of space. The degree of abstraction required for the formulation of the axioms of such a structure is decidedly greater than it was in the preceding examples; the character of the present article makes it necessary to refer interested readers to special treatises. See, for example, [5].

5. The standardization of mathematical technique. We have probably said enough to enable the reader to form a fairly accurate idea of the axiomatic method. It should be clear from what precedes that its most striking feature is to effect a considerable economy of thought. The "structures" are tools for the mathematician; as soon as he has recognized among the elements, which he is studying, relations which satisfy the axioms of a known type, he has at his disposal immediately the entire arsenal of general theorems which belong to the structures of that type. Previously, on the other hand, he was obliged to forge for himself the means of attack on his problems; their power depended on his personal talents and they were often loaded down with restrictive hypotheses, resulting from the peculiarities of the problem that was being studied. One could say that the axiomatic method is nothing but the "Taylor system" for mathematics.

This is however, a very poor analogy; the mathematician does not work like a machine, nor as the workingman on a moving belt; we can not over-emphasize the fundamental role played in his research by a special intuition,* which is not the popular sense-intuition, but rather a kind of direct divination (ahead of all reasoning) of the normal behavior, which he seems to have the right to expect of mathematical beings, with whom a long acquaintance has made him as familiar as with the beings of the real world. Now, each structure carries with it its own language, freighted with special intuitive references derived from the theories from which the axiomatic analysis described above has derived the structure. And, for the research worker who suddenly discovers this structure in the phenomena which he is studying, it is like a sudden modulation which orients at one stroke in an unexpected direction the intuitive course of his thought and which illumines with a new light the mathematical landscape in which he is moving about. Let us think—to take an old example—of the progress made at the beginning of the nineteenth century by the geometric representation of imaginaries. From our point of view, this amounted to discovering in the set of complex numbers a well-known topological structure, that of the Euclidean plane, with all the possibilities for applications which this in-

* Like all intuitions, this one also is frequently wrong.

volved; in the hands of Gauss, Abel, Cauchy and Riemann, it gave new life to analysis in less than a century. Such examples have occurred repeatedly during the last fifty years; Hilbert space, and more generally, functional spaces, establishing topological structures in sets whose elements are no longer points, but functions; the theory of the Hensel p-adic numbers, where, in a still more astounding way, topology invades a region which had been until then the domain *par excellence* of the discrete, of the discontinuous, *viz.* the set of whole numbers; Haar measure, which enlarged enormously the field of application of the concept of integral, and made possible a very profound analysis of the properties of continuous groups;—all of these are decisive instances of mathematical progress, of turning points at which a stroke of genius brought about a new orientation of a theory, by revealing the existence in it of a structure which did not *a priori* seem to play a part in it.

What all this amounts to is that mathematics has less than ever been reduced to a purely mechanical game of isolated formulas; more than ever does intuition dominate in the genesis of discoveries. But henceforth, it possesses the powerful tools furnished by the theory of the great types of structures; in a single view, it sweeps over immense domains, now unified by the axiomatic method, but which were formerly in a completely chaotic state.

6. A general survey. Let us now try, guided by the axiomatic concept, to look over the whole of the mathematical universe. It is clear that we shall no longer recognize the traditional order of things, which, just like the first nomenclatures of animal species, restricted itself to placing side by side the theories which showed greatest external similarity. In place of the sharply bounded compartments of algebra, of analysis, of the theory of numbers, and of geometry, we shall see, for example, that the theory of prime numbers is a close neighbor of the theory of algebraic curves, or, that Euclidean geometry borders on the theory of integral equations. The organizing principle will be the concept of a hierarchy of structures, going from the simple to the complex, from the general to the particular.

At the center of our universe are found the great types of structures, of which the principal ones were mentioned above; they might be called the mother-structures. A considerable diversity exists in each of these types; one has to distinguish between the most general structure of the type under consideration, with the smallest number of axioms, and those which are obtained by enriching the type with supplementary axioms, from each of which comes a harvest of new consequences. Thus, the theory of groups contains, beyond the general conclusions valid for all groups and depending only on the axioms enunciated above, a particular theory of finite groups (obtained by adding the axiom that the number of elements of the group is finite), a particular theory of abelian groups (in which $x \tau y = y \tau x$ for every x and y), as well as a theory of finite abelian groups (where these two axioms are supposed to hold simultaneously). Similarly, in the theory of ordered sets, one notices in particular those sets

(as for example, the set of integers, or of real numbers) in which any two elements are comparable, and which are called totally ordered. Among the latter, further attention is given to the sets which are called well-ordered (in which, as in the set of integers greater than 0, every subset has a "least element"). There is an analogous gradation among topological structures.

Beyond this first nucleus, appear the structures which might be called multiple structures. They involve two or more of the great mother-structures simultaneously not in simple juxtaposition (which would not produce anything new), but combined organically by one or more axioms which set up a connection between them. Thus, one has topological algebra. This is a study of structures in which occur at the same time, one or more laws of composition and a topology, connected by the condition that the algebraic operations be (for the topology under consideration) continuous functions of the elements on which they operate. Not less important is algebraic topology, in which certain sets of points in space, defined by topological properties (simplexes, cycles, etc.) are themselves taken as elements on which laws of composition operate. The combination of order structures and algebraic structures is also fertile in results, leading, in one direction to the theory of divisibility and of ideals, and in another to integration and to the "spectral theory" or operators, in which topology also joins in.

Farther along we come finally to the theories properly called particular. In these the elements of the sets under consideration, which, in the general structures have remained entirely indeterminate, obtain a more definitely characterized individuality. At this point we merge with the theories of classical mathematics, the analysis of functions of a real or complex variable, differential geometry, algebraic geometry, theory of numbers. But they have no longer their former autonomy; they have become crossroads, where several more general mathematical structures meet and react upon one another.

To maintain a correct perspective, we must at once add to this rapid sketch, the remark that it has to be looked upon as only a very rough approximation of the actual state of mathematics, as it exists; the sketch is schematic, and idealized as well as frozen.

Schematic—because in the actual procedures, things do not happen in as simple and as systematic a manner as has been described above. There occur, among other things, unexpected reverse movements, in which a specialized theory, such as the theory of real numbers, lends indispensable aid in the construction of a general theory like topology or integration.

Idealized—because it is far from true that in all fields of mathematics, the role of each of the great structures is clearly recognized and marked off; in certain theories (for example in the theory of numbers), there remain numerous isolated results, which it has thus far not been possible to classify, nor to connect in a satisfactory way with known structures.

Finally *frozen*,—for nothing is farther from the axiomatic method than a static conception of the science. We do not want to lead the reader to think that we claim to have traced out a definitive state of the science. The structures

are not immutable, neither in number nor in their essential contents. It is quite possible that the future development of mathematics may increase the number of fundamental structures, revealing the fruitfulness of new axioms, or of new combinations of axioms. We can look forward to important progress from the invention of structures, by considering the progress which has resulted from actually known structures. On the other hand, these are by no means finished edifices; it would indeed be very surprising if all the essence had already been extracted from their principles. Thus, with these indispensable qualifications, we can become better aware of the internal life of mathematics, of its unity as well as of its diversity. It is like a big city, whose outlying districts and suburbs encroach incessantly, and in a somewhat chaotic manner, on the surrounding country, while the center is rebuilt from time to time, each time in accordance with a more clearly conceived plan and a more majestic order, tearing down the old sections with their labyrinths of alleys, and projecting towards the periphery new avenues, more direct, broader and more commodious.

7. Return to the past and conclusion. The concept which we have tried to present in the above paragraphs, was not formed all at once; rather is it a stage in an evolution, which has been in progress for more than a half-century, and which has not escaped serious opposition, among philosophers as well as among mathematicians themselves. Many of the latter have been unwilling for a long time to see in axiomatics anything else than futile logical hairsplitting not capable of fructifying any theory whatever. This critical attitude can probably be accounted for by a purely historical accident. The first axiomatic treatments and those which caused the greatest stir (those of arithmetic by Dedekind and Peano, those of Euclidean geometry by Hilbert) dealt with univalent theories, *i.e.*, theories which are entirely determined by their complete system of axioms; for this reason they could not be applied to any theory except the one from which they had been extracted (quite contrary to what we have seen, for instance, for the theory of groups). If the same had been true for all other structures, the reproach of sterility brought against the axiomatic method, would have been fully justified.* But the further development of the method has revealed its power; and the repugnance which it still meets here and there, can only be explained by the natural difficulty of the mind to admit, in dealing with a concrete problem, that a form of intuition, which is not suggested directly by the given elements (and which often can be arrived at only by a higher and frequently difficult stage of abstraction), can turn out to be equally fruitful.

As concerns the objections of the philosophers, they are related to a domain, on which for reasons of inadequate competence we must guard ourselves from

* There also occurred, especially at the beginning of axiomatics, a whole crop of monster-structures, entirely without applications; their sole merit was that of showing the exact bearing of each axiom, by observing what happened if one omitted or changed it. There was of course a temptation to conclude that these were the only results that could be expected from the axiomatic method.

entering; the great problem of the relations between the empirical world and the mathematical world.* That there is an intimate connection between experimental phenomena and mathematical structures, seems to be fully confirmed in the most unexpected manner by the recent discoveries of contemporary physics. But we are completely ignorant as to the underlying reasons for this fact (supposing that one could indeed attribute a meaning to these words) and we shall perhaps always remain ignorant of them. There certainly is one observation which might lead the philosophers to greater circumspection on this point in the future: before the revolutionary developments of modern physics, a great deal of effort was spent on trying to derive mathematics from experimental truths, especially from immediate space intuitions. But, on the one hand, quantum physics has shown that this macroscopic intuition of reality covered microscopic phenomena of a totally different nature, connected with fields of mathematics which had certainly not been thought of for the purpose of applications to experimental science. And, on the other hand, the axiomatic method has shown that the "truths" from which it was hoped to develop mathematics, were but special aspects of general concepts, whose significance was not limited to these domains. Hence it turned out, after all was said and done, that this intimate connection, of which we were asked to admire the harmonious inner necessity, was nothing more than a fortuitous contact of two disciplines whose real connections are much more deeply hidden than could have been supposed *a priori*.

From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures; and it so happens—without our knowing why—that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation. Of course, it can not be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by deliberately throwing out this content, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations and for the development of their full power.

It is only in this sense of the word "form" that one can call the axiomatic method a "formalism." The unity which it gives to mathematics is not the armor of formal logic, the unity of a lifeless skeleton; it is the nutritive fluid of an organism at the height of its development, the supple and fertile research instrument to which all the great mathematical thinkers since Gauss have contributed, all those who, in the words of Lejeune-Dirichlet, have always labored to "substitute ideas for calculations."

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* We do not consider here the objections which have arisen from the application of the rules of formal logic to the reasoning in axiomatic theories; these are connected with logical difficulties encountered in the theory of sets. Suffice it to point out that these difficulties can be overcome in a way which leaves neither the slightest qualms nor any doubt as to the correctness of the reasoning; [2] and [3] are valuable references for this point.

References

1. L. Brunschvicg, Les étapes de la philosophie mathématique, Paris, Alcan, 1912.
2. J. Dieudonne, Les méthodes axiomatiques modernes et les fondements des mathématiques, Revue Scientifique, LXXVII, 1939, pp. 224–232.
3. H. Cartan, Sur le fondement logique des mathématiques, Revue Scientifique, LXXXI, 1943, pp. 3–11.
4. N. Bourbaki, Eléments de mathématique, book I (fasc. de résultats), Actual. Scient. et Industr., no. 846.
5. ———, Eléments de mathématique, book III, introduction and chapter I, Actual. Scient. et Industr., no. 858.

THE GEOMETRIES OF THE THERMAL AND GRAVITATIONAL FIELDS*

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1. Introduction. In physics, geometry is—or should be—what you measure it. Deviations from the geometry or kinematics you expect to find, on *a priori* or other grounds, may lead to the discovery or better understanding of some hitherto overlooked element in the realm under examination. To such deviations may be attributed the discovery by the ancients of the curvature of the earth's surface, by Kepler of the laws of planetary motion, and by Einstein of the special and general theories of relativity. Perhaps the most spectacular in its impact on the modern mind, because we are still immersed in the intellectual milieu in which it arose, is Einstein's geometrization of the universal force of gravitation. Here the recognition of the appropriateness of a certain Riemannian kinematics made possible the incorporation of gravitation into the geometrical structure of space-time, and by identification of inertial and gravitational mass made tautological the law which states their equivalence.

To illustrate the relationship between geometry and physical law, I have elected here first to discuss in some detail the problem of the geometry exhibited by a "hot plate." We shall find, as could be foreseen, that the geometry disclosed by measurement depends not on the state of the plate alone, but also on the method of measurement, *e.g.*, whether by purely optical means or by the use of measuring rods which are brought into thermal equilibrium with the plate before being read. The arbitrariness (within, however, fairly closely prescribed limits) of the geometry thus found will support us in the common-sense view that the plate is actually flat, and that the observed deviations are to be attributed to quite well understood forces (thermal stresses) which act differ-

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entially on the various instruments which may be employed in the measurement.

Leaving the rather prosaic domain of common sense, we next consider the extension of this problem to the 3-dimensional thermal field. Here it will be suggested that the physical laws of heat conduction may (at least approximately) be given a purely geometrical formulation. But again the strong dependence of the geometry thus disclosed on the particular method of measurement will prevent us from taking our proposed theory too seriously.

Finally, we shall contrast the geometry of the thermal field with the geometry of the gravitational field in order to see why gravitational, as opposed to thermal, forces may satisfactorily be incorporated into the geometrical structure of space and time. This comparison will be carried out in terms of a sort of special relativity caricature of the general relativity theory of gravitation, which bears a striking formal relationship to the thermal geometry—and which, I hold, presents the elements of the Einstein theory necessary to our purposes without involving us in the mathematical intricacies of the full theory.*

2. The hot plate. Let us suppose we have before us a flat sheet of metal, of uniform thickness t, and that we can by judicious use of heat sources, constant temperature baths and the like produce on the plate any temperature distribution consistent with the laws of heat conduction. Let us further suppose the plate is so constrained that it cannot stretch or buckle, despite the thermal stresses induced in it by the treatment, so that it can be said to remain flat by ordinary laboratory criteria.

Confining our attention to stationary states, and considering heat losses from the free faces of the plate as negligible, the temperature $T(x, y)$ at any point $P(x, y)$ of the plate is governed by the equation

$$k\nabla^2 T + p = 0, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{1.1}$$

k is the thermal conductivity of the plate material and $p(x, y)$ is the source density of heat at $P(x, y)$, *i.e.*, $ptdxdy$ is the rate at which heat is being supplied to the plate through an area $dxdy$ about P. Heat flows past P in the direction of the negative gradient $-\nabla T$ of T, and the rate of flow across unit length normal to this direction is t times the magnitude of the flux vector

$$\mathbf{f} = -k\nabla T. \tag{1.11}$$

We propose now to consider the geometry of the plate as revealed by measurements made as follows: a short graduated metal rod, made of a material of thermal coefficient of expansion h, is placed upon the plate, and distances are measured *after the rod has come into thermal equilibrium with the plate* in the neighborhood of the point $P(x, y)$ of interest. But then a linear element of

* For a more searching account of the epistemological background of the present paper see the author's chapter "Geometry as a Branch of Physics" in the forthcoming volume, Albert Einstein: Philosopher-Scientist, of The Library of Living Philosophers.

"true" length ds going out from the point $P(x, y)$ on the plate will be measured as having a length

$$d\sigma = \frac{ds}{1 + hT(x, y)};$$

in terms of Cartesian coordinates x, y the linear element

$$d\sigma^2 = \frac{dx^2 + dy^2}{[1 + hT(x, y)]^2}. \tag{1.2}$$

Now this linear element (1.2) may (but need not) be considered as defining a 2-dimensional curved surface, in terms of the "isothermal" coordinates x, y. The Gaussian or total curvature K of this hypothetical surface is readily computed to be*

$$K = h(1 + hT)\nabla^2 T - h^2(\nabla T)^2, \tag{1.3}$$

where ∇T is the ordinary gradient $(\partial T/\partial x, \partial T/\partial y)$ of T, and $(\nabla T)^2$ is the square of its magnitude in the Euclidean sense. On taking the physical laws expressed by equations (1.1), (1.11) into account, the Gaussian curvature of the hot plate, as implied by the prescribed measuring procedure, is

$$K = -\frac{h}{k}(1 + hT)p - \left(\frac{h}{k}\right)^2 f^2. \tag{1.31}$$

From this it follows that in any region in which no heat is extracted from the plate ($p \geqq 0$) the "curvature" of the plate is negative; if in particular $p = 0$

$$K = -\left(\frac{h}{k} f\right)^2 \tag{1.32}$$

and the quantity

$$R = |K|^{-1/2} = \frac{k}{h|f|}, \tag{1.33}$$

which has the physical dimensions of a length and may conveniently be called the "radius of curvature" of the plate at $P(x, y)$, is inversely proportional to the heat flux through P. The case in which $p = 0$ and the curvature is constant over the whole plate leads to the hyperbolic geometry of the Poincaré half-plane; this case has been given a very elegant and detailed treatment in this MONTHLY by E. W. Barankin.**

Returning to the general case, the lines which play the role of the straights of euclidean geometry are the geodesics, defined as the extremals $y = F(x)$ of the

* Cf. L. P. Eisenhart's Introduction to Differential Geometry, p. 154 (Princeton 1940); quoted hereafter as IDG.

** "Heat Flow and Non-Euclidean Geometry," this MONTHLY vol. 49, 1942, pp. 4–14.

variation principle

$$\delta \int d\sigma \equiv \delta \int \frac{(1 + F'^2)^{1/2}}{1 + hT(x, F)}\, dx = 0. \tag{1.4}$$

These geodesics are convex toward the side on which T increases, as can be seen intuitively from the fact that for sufficiently short geodesic arcs the integral in (1.4) above must be minimal, and since the measuring rod expands more on the side toward which T is increasing, the measured distance along an arc bowed slightly in this direction will be less than along the chord joining its endpoints. From the standpoint of the measured geometry the Euclidean straight

$$x \sin a - y \cos a = \text{const.}$$

of slope tan a is actually "curved"; its "geodesic curvature" is found to be*

$$\kappa_g = h(\nabla T)_n = -\frac{h}{k} f_n,$$

where $f_n = f_y \cos a - f_x \sin a$ is the heat flux across unit length of the straight. We have here a geometrical interpretation of the flux of heat in terms of the geodesic curvature of the Euclidean straights; the components $-f_y$, f_x of the flux, rotated through $\pi/2$, are proportional to the geodesic curvature of the lines of parameter x, y, respectively, through the point P of interest.

We have thus shown that the geometry of the heated plate is to some extent arbitrary, depending as it does on the coefficient of expansion h of the measuring instrument, and is, in particular, flat for an ideal rod for which $h = 0$. The geometry found is in fact interpretable as that of a curved surface, of curvature defined by Equation (1.3). In order to gain some appreciation of the magnitudes involved, consider the case of a steel plate ($k = 0.11$ cal/cm sec deg C) measured with a steel rod ($h = 0.11 \times 10^{-4}$/deg C). At a point P at which the flux f is 1 cal/cm^2 sec the Gaussian curvature K of the plate is -10^{-8} cm^{-2}, or its equivalent radius of curvature $R = 10^4$ cm$\sim$320 ft; the maximum geodesic curvature of an Euclidean straight through P is then 10^{-4} cm^{-1}, or the radius of geodesic curvature of the straight is 10^4 cm.

3. The thermal field in space. We turn next to the generalization of the above problem to 3-dimensional space, where our predilection for considering the measured space as imbedded in a higher-dimensional flat space will surely lose some of its appeal. Imagine, then, a material of heat conductivity k filling some part of our flat physical space, which is described by cartesian coordinates $(x_i) = (x, y, z)$. Suppose, by some stretch of the imagination, that it is possible to explore this thermal material with a short measuring rod, of thermal coefficient of expansion h. The temperature $T(xyz)$ at any point P(xyz) in the medium will again be governed by the fundamental equations (1.1), (1.11), where ∇ is

* IDG, p. 186.

now the 3-dimensional gradient operator $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ and $p(xyz)$ is the source strength per unit volume at P.

The measured geometry at the thermal field will then be defined by the linear element

$$d\sigma^2 = \frac{dx^2 + dy^2 + dz^2}{[1 + hT(xyz)]^2}. \tag{2.1}$$

The geodesics $\delta\int d\sigma = 0$ are now space-curves, with respect to which an euclidean straight has a point P of it a geodesic curvature

$$\kappa_g = h(\nabla T)_n = -\frac{h}{k} f_n, \tag{2.2}$$

where f_n is the component of the flux at P normal to the given straight.

The intrinsic geometry of such a 3-dimensional Riemannian space is no longer, as in the 2-dimensional case treated above, characterized by a single scalar curvature K. There is instead at each point P a curvature $K(\mathrm{P}, \mathbf{a})$ associated with each direction $\mathbf{a} = (a_i)$ through P, whose geometrical interpretation is as follows: Construct through P the 1-parameter family of geodesics orthogonal to the given direction $\mathbf{a}$, thus forming a 2-dimensional surface through P; then $K(\mathrm{P}, \mathbf{a})$ is the Gaussian curvature of this surface at P. On taking the direction coefficients a_i as normalized in the euclidean sense $\sum a_i^2 = 1$, the directional curvature of the space defined by the linear element (2.1) is*

$$K(\mathrm{P}, \mathbf{a}) = h(1 + hT)[\nabla^2 T - \sum T_{ij} a_i a_j] - h^2(\nabla T)^2, \tag{2.3}$$

where $T_{ij} = \partial^2 T/\partial x_i \partial x_j$. The directional quality of this curvature is here due to its dependence on the second derivative

$$\frac{d^2T}{dn^2} = \sum T_{ij} a_i a_j$$

of the temperature along the normal $\mathbf{a}$ to the geodesic surface of which it is the Gaussian curvature.

It will be most convenient for our purposes to introduce here an intrinsic "mean curvature" $\overline{K}(\mathrm{P})$ which is independent of direction, defined as the mean of the directional curvatures (2.3) associated with any triplet $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of three mutually perpendicular directions. It follows that

$$\overline{K}(P) = \tfrac{2}{3} h(1 + hT)\nabla^2 T - h^2(\nabla T)^2, \tag{2.31}$$

as may most readily be seen by computing the mean in that Cartesian system whose axes are in the directions defined by the triplet; $\overline{K}$ is accordingly a scalar, independent of the particular triplet chosen.

* As may be computed as in L. P. Eisenhart's Riemannian Geometry, p. 79 (Princeton 1926). These directional curvatures are expressible in terms of the components a_i and the six linearly independent components of the Riemann-Christoffel tensor R_{ijkl}, or the equivalent six components of the Ricci tensor $R_{jk} \equiv R^i_{jki}$. The intrinsic mean curvature $\overline{K}$ introduced below is equal to $-1/6$ the scalar contraction R^i_i of the Ricci tensor.

A geometrical interpretation of the physical laws (1.1), (1.11) of heat conduction can now be obtained in terms of the mean curvature $\overline{K}$, whose value is here found to be

$$\overline{K} = -\frac{2h}{3k}(1 + hT)p - \left(\frac{h}{k}\right)^2 f^2. \tag{2.32}$$

In source-free regions this reduces to

$$\overline{K} = -\left(\frac{h}{k}\right)^2 f^2; \tag{2.33}$$

that is, the mean curvature at such a point P is, as in the 2-dimensional case (1.32), negative and proportional to the square of the heat flux f past P. And as in the previous case, the deviation from flatness in the measured geometry can, in principle, be determined by intrinsic measurements alone, without reference to any hypothetical imbedding space. Thus the surface S and the volume V of a "sphere" of measured (geodesic) radius ρ, *i.e.*, the locus of all points at a given geodesic distance ρ from P, differ from their flat values $4\pi\rho^2$, $4\pi\rho^3/3$ by amounts which depend on the intrinsic curvature at points within the sphere. For a small sphere of radius $\rho \ll 1/|\overline{K}|^{1/2}$ these measures are given by expressions whose leading terms are

$$\begin{cases} S = 4\pi\rho^2(1 - \overline{K}\rho^2/3 + \cdots), \\ V = \frac{4}{3}\pi\rho^3(1 - \overline{K}\rho^2/5 + \cdots), \end{cases} \tag{2.4}$$

where $\overline{K}$ is the value of the mean curvature at the center P of the sphere; to the order here considered these expressions are identical with those for a sphere of geodesic radius ρ in a spherical or hyperbolic 3-space of constant curvature $\overline{K}$.

As an example of the type of geometry to which one is led, consider the case of a spherically symmetric thermal field generated by a point source of heat of strength P cal/sec at the origin $r=0$. The temperature resulting at a cartesian distance r from the point source is

$$T = \frac{P}{4\pi kr}, \tag{2.5}$$

and the mean curvature there is

$$\overline{K} = -\frac{a^2}{r^4}, \quad \text{where} \quad a = \frac{hP}{4\pi k}. \tag{2.51}$$

It may be noted that, because of the symmetry, any geodesic in this field is a "plane" curve, in the sense that it must lie completely in a cartesian plane, such as $y=0$, passing through the origin. But the curvature of this geodesic meridian surface, as implied by the measuring technique, is found to be a/r^3, and the mean curvature $\overline{K}$ at P$(r, 0, 0)$ is the mean of the curvature a/r^3 of two such

meridians $y=0$, $z=0$ and the curvature $-2a/r^3-3a^2/r^4$ of the surface generated by all geodesics tangent at P to the cartesian plane $x=$const.

A more interesting case, because it leads to the classical non-euclidean geometries, is that in which there is a spherical region of radius r_1 at all points of which heat is supplied to the medium at the constant rate of p cal/cm^3 sec. The temperature at any point at cartesian distance r from the source is found, on integrating the heat conduction equation (1.1), to be

$$T=\begin{cases} -\dfrac{p}{6k}r^2, & 0\leqq r\leqq r_1, \\ -\dfrac{p}{3k}r_1^2\left(\dfrac{3}{2}-\dfrac{r_1}{r}\right), & r_1\leqq r, \end{cases} \tag{2.6}$$

where the constant of integration is so chosen that $T=0$ at the center. Note that the outside field is, as it should be, that for a point source of strength $P=4\pi r_1^3 p/3$.

Now within the sphere $r=r_1$, the mean curvature $\overline{K}$ and radius of curvature $\overline{R}$ have the constant values

$$\overline{K}=-\frac{2}{3}\frac{hp}{k}, \qquad \overline{R}=\frac{1}{|\overline{K}|^{1/2}}=\left|\frac{3k}{2hp}\right|^{1/2}. \tag{2.61}$$

But also within the source the directiona' curvature $K(\mathrm{P}, \mathbf{a})$ defined by equation (2.3) is constant and equal to $\overline{K}$, and is accordingly independent of both position and orientation. The geometry revealed by measurement within the cartesian sphere $r=r_1$ is therefore one of the classical non-euclidean geometries, the hyperbolic geometry of Bolyai and Lobachewski if the central region is a source ($p>0$, whence $K<0$), and the spherical geometry of Riemann if it is a sink ($p<0$, whence $K>0$).

This circumstance merits some further examination. In either case the measured geometry is a true non-homaloidal congruence geometry; each configuration of point, incident direction and incident 2-spread of directions containing the given direction, is equivalent to every other such configuration, and may be transformed into it by a motion of the space. The case $p>0$ has been cited by Poincaré in illustration of the 3-dimensional hyperbolic geometry, for which it offers a finite euclidean model.* For let us suppose that the cartesian radius of the source region is

$$r_1=\left(\frac{6k}{hp}\right)^{1/2}=2R$$

(which for unit source-strength $p=1$ in the example $h/k=10^{-4}$ considered above, is around 245 cm), then at the boundary $r=r_1$ of the source the rule will have shrunk to zero length, admitting such an outrageous extrapolation of the linear

* H. Poincaré, Science and Hypothesis, pp. 76–78 (Science Press 1929).

expansion law! The measured radius of the source will in consequence become infinite, as can be seen by considering the integral of $d\sigma$ between $r=0$ and $r=r_1$. The cartesian sphere $r \leqq r_1$, supplied with the metric

$$d\sigma^2 = \frac{dx^2 + dy^2 + dz^2}{[1 - (r/r_1)^2]^2},$$

thus presents a finite model of the open hyperbolic geometry, in which by its own measuring standards and procedures the volume is infinite and in which every full geodesic is of infinite length.

The case $p<0$, in which there is a spherical region of radius r_1 within which heat is abstracted from the medium at the constant rate p cal/cm³ sec, offers on the other hand a model of a finite space of constant positive curvature which is represented by the entire flat cartesian space. For here the measured radius of the sink is

$$\rho_1 = \int_0^{r_1} \frac{dr}{1 + r^2/4R^2} = 2R \tan^{-1} \frac{r_1}{2R},$$

and this approaches the finite limit πR as $r_1 \to \infty$. The measured length of the full Euclidean straight, and therefore of every other full geodesic of the space, is thus $2\pi R$.

4. Geometrization of the physical law. The results obtained in the above sections present a possible geometrical interpretation of the physical laws of heat conduction. Thus in a source-free region the mean curvature $\overline{K}$ at a point P is a measure of the heat flux past P, and the direction of flow is that of a cartesian straight whose geodesic curvature κ_g is zero, as follows from equations (2.33), (2.2). So far as the temperature is concerned, we may think of it as prescribed either by

(a) the physical law (1.1),

$$k\nabla^2 T + p = 0,$$

with appropriate boundary conditions, or by

(b) the requirement that the mean curvature [eq. (2.32)]

$$\overline{K} = -\frac{2h}{3k}(1 + hT)p - \left(\frac{h}{k}\right)^2 f^2.$$

This second viewpoint is, however, but a clumsy circumlocution of the first; it is a geometrical interpretation of the physical law, but in no way a modification of it.

A far more complete and satisfactory geometrization of the physical law would be possible had it turned out that the mean curvature $\overline{K}(P)$ of the medium, under the adopted measuring prescription, were determined directly in terms of the given source distribution $p(P)$, without involving explicitly either

T or its gradient, as is in fact the case for a uniform source distribution, equation (2.61). And because it is by following this line of attack that the general theory of relativity can be evolved from the classical theory of gravitation, we shall consider the corresponding modification of the theory of heat conduction. It is to be expected that here, as well as there, the resulting geometrized theory will constitute a factual modification, rather than a mere geometrical interpretation, of the "classical" theory.

We note that for sufficiently weak thermal fields, say in which the temperature differences are such that $h\Delta T \ll 1$, the first term $-2hp/3k$ in the mean curvature $\overline{K}$, eq. (2.32), is predominant; the remaining two terms are, in regions in which heat is being supplied, of the relative order $h\Delta T$. Under these conditions the mean curvature is given approximately by

$$\overline{K} = -\frac{2h}{3k}p; \tag{3.1}$$

we propose to effect the complete geometrization sought above by replacing the "classical" equation (1.1) *by the equation* (3.1), which requires that the mean curvature $\overline{K}$ of space at any point P be rigorously proportional to the strength p of the source density at P. That this geometrization is a factual, and not merely a formal, modification of the equation of heat conduction is apparent on noting that the mean curvature $\overline{K}$ at a source-free point P vanishes, and is not, as in equation (2.33), a measure of the heat flux past P.

It will now be found convenient to replace T by the dependent variable U defined by

$$1 + hT = U^{-2}, \tag{3.2}$$

i.e., to take the linear element (2.1) of the space in the form

$$d\sigma^2 = U^4(dx^2 + dy^2 + dz^2). \tag{3.21}$$

The mean curvature is then

$$\overline{K} = -\frac{4}{3}U^{-5}\nabla^2 U, \tag{3.22}$$

and the proposed field equation (3.1) for heat conduction

$$U^{-5}\nabla^2 U = \frac{h}{2k}p. \tag{3.23}$$

The case of an isotropic point-source of heat will illustrate the kind of deviations from the "classical" theory, developed in Section 3 above, which are here to be expected. In this case $p=0$ except at the origin, and the general solution of equation (3.23) satisfying this condition is

$$U = A + \frac{B}{r}. \tag{3.3}$$

The natural normalization $U=1$ at $r=\infty$ requires that $A=1$, and the approximate correspondence with the classical theory in regions $r \gg 1$ of weak fields that $B=-hP/8\pi k$ where P is the strength of the source. The linear element of the physical geometry is then

$$d\sigma^2 = \left(1 - \frac{a}{2r}\right)^4 [dx^2 + dy^2 + dz^2], \tag{3.31}$$

where a is again $hP/4\pi k$.

We have here an amusing coincidence which is worthy of note: the spatial geometry defined by this $d\sigma^2$ is precisely that of the spatial section $t=$const. of a point-mass

$$m = -ac^2/G \tag{3.32}$$

in the general relativity theory of gravitation.* Here a sink of unit strength $P=-1$ cal/sec corresponds, in the example $h/k=10^{-4}$ cm sec/cal contemplated above, to a mass $m=10^{23}$ gm, *i.e.*, the thermal distortion caused by a sink of unit strength in steel is geometrically equivalent to the gravitational distortion caused by an asteroid some 100 miles in radius! Also, the singularity of $d\sigma^2$ at $r=a/2$ ($\sim 4\times 10^{-6}$ cm for a unit source) is at least superficially analogous to the well-known "Schwarzschild singularity" of the gravitational field.

Since the "classical" solution (2.6) inside a spherical region of constant source density p satisfies the new field equation (3.1), it may be taken over bodily into the proposed geometrized theory. An exterior solution of the form (3.31) may then be fitted on to represent the field outside the source; but now, because of the non-linearity of the new field equations, the total strength P of the equivalent point-source will no longer be equal to the spatial integral of the distributed strength p.

Glancing back at the 2-dimensional thermal field discussed in Section 1 above, we see from equation (1.31) that a corresponding geometrization could be obtained by replacing the physical law (1.1) by the requirement that the Gaussian curvature

$$\overline{K} = -\frac{h}{k} p. \tag{3.4}$$

But then in source-free regions $K=0$, and hence the surface is there developable. On making the change of variable.

$$1 + hT = e^{hU} \tag{3.41}$$

* Cf. A. S. Eddington's The Mathematical Theory of Relativity, p. 93 (Cambridge 1923).

the linear element (1.2) becomes

$$d\sigma^2 = e^{-2hU}(dx^2 + dy^2) \tag{3.42}$$

and the fundamental field equation (3.4)

$$k\nabla^2 U + e^{-2hU}p = 0. \tag{3.43}$$

In source-free regions, where $\nabla^2 U = 0$, it follows that the developable surface is obtained by a conformal transformation $x, y \rightarrow \xi, \eta$, and that the geodesics of the measured geometry are those curves which map into the straights of the ξ, η plane; conformal mapping by elementary functions of a complex variable produces here many curious and amusing examples of physical geometry. But we here confine ourselves to the remark that, as may be seen from equation (3.42), the measured geometry is equivalent to that which would result from the "classical" conduction theory on postulating that the expansion δl of a rod of length l due to an increase δT in temperature is given by $\delta l/l = h\delta T$, instead of by $\delta l/l_0$ as in Sections 1–2 above.

Returning to the 3-dimensional case, we see that we have succeeded in geometrizing the theory of heat conduction, in the sense that the content of the fundamental equation (1.1), somewhat modified, is contained in the purely geometrical assertion (3.1) that the mean curvature $\overline{K}$ of the thermal medium is proportional to the source-density p. But the geometry thus obtained is strongly dependent upon the measuring instrument, for the constant of proportionality has in it the ratio h/k of the coefficient of expansion h of the material of which the measuring rod is made, as well as the coefficient of heat conduction k of the thermal medium; thus were we to employ a brass rod, instead of a steel one, we would find that the curvature is trebled. The proposed theory of heat conduction is in consequence a poor example of a geometrico-physical theory, for the geometry obtained lacks the desirable quality of universality. We are therefore better advised to consider the geometry of the thermal field as Euclidean, and to attribute measured deviations from flatness to thermal stresses, differing from material to material, set up in the measuring rods.

This geometrical theory, in fine, would have far more appeal if the ratio h/k were a universal constant, for then the geometry implied by equation (2.32)or its modification (3.1) would be independent of the choice of measuring rod. And the analogue of this is precisely what happens in the general theory of relativity, to a caricature of which we now turn our attention.

5. The gravitational field. To consider the geometry of the gravitational field, it is desirable to proceed immediately to a 4-dimensional formulation, for even though we propose to deal with static fields the most appropriate method of exploration will be dynamical—namely, by examining the motion of test particles set loose in the field. In order to avoid the full mathematical complexity of the general theory of relativity we here set up a theory of gravitation which can be incorporated into the special theory of relativity, and which bears a striking resemblance to the theory of heat conduction presented above.

The kinematical background of this theory is given by the Minkowski linear element

$$ds^2 = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2), \tag{4.1}$$

and the field equation to be geometrized is taken to be the fairly immediate special relativistic generalization

$$\Box^2 V = -4\pi G c^2 \rho \tag{4.11}$$

of Poisson's equation for the gravitational potential V due to a mass distribution of density ρ; here $\Box$ is the 4-dimensional gradient operator whose square

$$\Box^2 = \frac{\partial^2}{\partial t^2} - c^2\nabla^2.$$

A test particle of gravitational mass m_G will be acted upon by a force determined by the contravariant vector

$$m_G f^i = m_G\left(\frac{1}{c^2}\frac{\partial V}{\partial t}, -\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right), \tag{4.12}$$

but since its proper inertial mass m_I is to be constant along the trajectory, only the component of this vector orthogonal to the world-line is effective. This leads to the unique equations of motion

$$m_I \frac{d^2x^i}{ds^2} = m_G\left(f^i - \sum_j f_j \frac{dx^j}{ds}\frac{dx^i}{ds}\right).$$

It is now easy to show that the trajectories (4.13) are in fact the geodesics of the 4-dimensional space-time defined by the linear element

$$d\sigma^2 \equiv \exp\left(\frac{2m_G V}{m_I c^2}\right) ds^2; \tag{4.2}$$

we have thereby accomplished a thorough geometrization of the dynamical trajectories. It would at first sight appear that this geometry depends upon the test particle as well as upon the field, for the ratio m_G/m_I of its gravitational and inertial mass enters explicitly into the linear element. But this is only apparent, for it has been known empirically since Newton that these two masses are directly proportional; on measuring them in the same unit the ratio becomes unity, and the linear element (4.2) may henceforth be taken in the form

$$d\sigma^2 = e^{2V/c^2} ds^2 \tag{4.21}$$

from which all trace of the exploring particle has been removed.*

* The variation principle $\delta\int d\sigma = 0$ defining the geodesics of (4.21) goes over, on allowing $c \to \infty$, into Hamilton's principle $\delta\int(T - V)dt = 0$ of classical mechanics.

This form suggests, in analogy with the development in Sections 1–2 above, that we might base this linear element on the postulate that a measuring rod of length l suffers a fractional change of length $\delta l/l = -\delta V/c^2$ on transportation to a position at which the potential energy V is increased by δV; our time standards must be supposed to suffer the same relative change, in order that the velocity of light c be unaltered. For then the length l of the rod at a position of potential V would be $l = l_0 e^{-V/c^2}$, where l_0 is its length at infinity ($V=0$), and the measurement of ds with such a standard would yield the value $d\sigma$ given by Equation (4.21). On introducing the new dependent variable $U = e^{V/c^2}$ suggested thereby, the linear element becomes

$$d\sigma^2 = U^2\left[dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)\right]. \tag{4.22}$$

In order to carry through the geometrization of the theory, we next examine the geometrical consequences of the given field equation (4.11). Proceeding as in the previous sections, we find that the mean curvature $\overline{K}$ of the 4-dimensional space-time (4.21), defined in fairly straight-forward generalization of the considerations given above for a 3-dimensional space, is

$$\overline{K} = -\frac{1}{2c^2} e^{-2V/c^2}\left[\Box^2 V + \frac{1}{c^2}(\Box V)^2\right]. \tag{4.23}$$

On taking the field equation (4.11) and the definition of force (4.12) into account,

$$\overline{K} = e^{-2V/c^2}[2\pi G\rho - \tfrac{1}{2}f^2]. \tag{4.24}$$

We propose now to replace the above theory by a geometrical one in which the geometry of space-time is related directly to the material content. The choice of field equation which naturally suggests itself from the form of equation (4.24) is

$$\overline{K} = 2\pi G\rho; \tag{4.3}$$

this equation differs from (4.11) only by terms of order $2V/c^2$, and leads in the dynamical applications to deviations of the order $(v/c)^2$ where v is the velocity of the test particle. We have then here a geometrical theory of gravitation which differs from the special relativistic theory set up above, and therefore from the classical Newtonian theory, only by terms of relative order $(v/c)^2$.

This theory is in principle very similar to the general relativity theory of gravitation, only far simpler, and, in the last analysis, far less satisfactory! Here we have a single scalar field equation (4.3) which expresses the mean curvature $\overline{K}$ of space-time directly in terms of the density of its material content; in the general theory the tensor field equations have 10 components, which express certain partial curvatures (the 10 components R_{ij} of the Ricci tensor)

in terms of the density, pressure and state of motion of the material content.* In the latter it is clear from the development that the density which enters directly into the geometry is the density of *inertial* mass; following this hint we postulate that it is this inertial density ρ which appears in equation (4.3).

On taking the linear element in the form (4.22) we find

$$\overline{K} = -\tfrac{1}{2}U^{-3}\Box^2 U = 2\pi G\rho; \tag{4.31}$$

this field equation reduces for the case of a static field to

$$U^{-3}\nabla^2 U = \frac{4\pi G}{c^2}\rho. \tag{4.32}$$

The solution of this equation for a point-mass m is

$$U = e^{V/c^2} = 1 - \frac{\mu}{r}, \tag{4.33}$$

where $\mu = Gm/c^2$ is the length-equivalent of the mass m. Taking the motion of a test particle as along a geodesic of the space-time (4.22), we readily find that in the classical approximation $(v/c)^2 \ll 1$ the motion is Newtonian. It turns out that in the next approximation the perihelion of an elliptic orbit suffers a retardation, the magnitude of which is 1/6 that of the Schwarzschild advance, and that a ray of light is not bent on passing a massive body. As may be seen from the measuring procedure suggested above, light going from a position of lower to one of higher gravitational potential suffers the red-shift $\Delta\lambda/\lambda = G\Delta v/c^2$, as in the Einstein theory. But such refinements are beyond our present interest, for our simple theory is on other, more weighty, counts quite inferior to the general theory of relativity.†

What we have shown here is that the gravitational field, as opposed to the thermal, can be given a quite satisfactory geometrization. The reason for this is that in the gravitational case the geometry arrived at is universal in virtue of the equality of the inertial and gravitational masses of the test particle employed in its exploration, whereas in the thermal the geometry is dependent on the physical properties of the measuring instrument. With this we have completed our announced task—a study of the physical geometry of the gravitational and of the thermal fields, and of the reasons for the success of the former, as opposed to the latter, as a basis for a tenable physical theory.

* In the general theory the "mean curvature" here employed (which is in fact $-1/12$ the contracted Ricci tensor R_i^i is determined by an equation of exactly the same form as (4.3) but with a numerical coefficient $-2\pi/3$ instead of $+2\pi$.

† Thus in the general theory the material-energetic conservation laws and the mechanical equations of motion are direct consequences of the field equations, whereas here they must be postulated independently.

MATHEMATICAL NOTES

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EXTENSION OF A TOPOLOGICAL PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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1. Introduction. In the August–September (1949) issue of this MONTHLY, a topological proof of the fundamental theorem of algebra was given by B. H. Arnold [1]. We here extend this proof to the case of an equation in quaternions, a result established earlier by Eilenberg and Niven [2]. It is of interest to note that of all the proofs of the fundamental theorem of algebra, only those using topological notions appear to be extensible to the case of quaternions. The well-known analytic proof [3] employing the Liouville theorem on bounded analytic functions does not seem to carry over to the more general case of quaternions. The theory of functions over non-commutative algebras [4] is, of course, not as well developed as that over complex numbers. Similarly the proof (Gauss' second) as given by MacDuffee [5] and van der Waerden [6] involves an algebraic technique which also does not appear to admit generalization.

2. Result. Let $f(z)$ be a polynomial of degree n of the type

$$f(z) = za_1za_2z \cdots za_n + \phi(z), \tag{1}$$

where $z, a_1, a_2, \cdots, a_n$ are real quaternions with $a_i \neq 0$ for $i = 1, 2, \cdots, n$, and where $\phi(z)$ is a sum of a finite number of similar monomials each of degree $< n$. We prove that the equation $f(z) = 0$ has at least one quarternion solution. The hypothesis that there is only one term of highest degree in $f(z)$ is necessary, as can be seen from such an example as $zj - iz - 1 = 0$. There is no loss of generality in assuming that the term of degree n in $f(z)$ has left coefficient unity, as indicated in (1); in fact we can also assume that

$$|a_1a_2 \cdots a_n| = 1. \tag{2}$$

3. Proof. Any non-zero quaternion $u = u_1 + u_2i + u_3j + u_4k$ can be written in the form

$$u = |u|(\cos\theta + \epsilon \sin\theta), \tag{3}$$

where $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$,

$$\cos\theta = \frac{u_1}{|u|}, \qquad \sin\theta = \frac{\sqrt{u_2^2 + u_3^2 + u_4^2}}{|u|}, \tag{4}$$

and ϵ is the unit vector,

$$\epsilon = \frac{u_2 i + u_3 j + u_4 k}{\sqrt{u_2^2 + u_3^2 + u_4^2}}. \tag{5}$$

If u is real, then ϵ is arbitrary, say $\epsilon = 1$, so that θ and $\epsilon \sin\theta$ are continuous functions of u, although ϵ itself is not a continuous function of u.

Let there be r terms in $\phi(z)$ in (1), say $\phi(z) = \phi_1(z) + \phi_2(z) + \cdots + \phi_r(z)$, and write $|\phi_m(1)| = b_m$ for $m = 1, 2, 3, \cdots, r$. Thus

$$|\phi_m(z)| \leqq b_m \qquad \text{for } |z| \leqq 1. \tag{6}$$

Then we define R and u by

$$R = 2 + \sum_{m=1}^{r} b_m, \qquad u = a_1 z a_2 z \cdots z a_n.$$

The mapping which we use is given by

$$g(z) = z - f(z)u^{-1}R^{-1}, \qquad \text{for } |z| \geqq 1; \tag{7}$$

$$g(z) = z - f(z)[\cos(|z|\theta) + \epsilon \sin(|z|\theta)]^{-1}R^{-1}, \qquad \text{for } 0 < |z| \leqq 1; \tag{8}$$

$$g(0) = -f(0)R^{-1}. \tag{9}$$

Now θ and ϵ are given by (3), (4), (5) so that u, θ and $\epsilon \sin\theta$ are continuous functions of z, except at $z = 0$. But $g(z)$ is a continuous function for all values of z, for the following reasons. First, (7) and (8) are continuous in the range indicated. Second, (7) and (8) are identical for $|z| = 1$. Third, the limit of $g(z)$ in (8) as $z \to 0$ exists and the limit is indicated in (9).

The proof that $|g(z)| \leqq R$ for $|z| \leqq R$ is now almost identical with Arnold's. For $|z| \leqq 1$ we have, by use of (2) and (6),

$$|g(z)| \leqq |z| + |f(z)| \cdot R^{-1} \leqq 1 + (1 + b_1 + b_2 + \cdots + b_r)R^{-1} \leqq R.$$

For $1 \leqq |z| \leqq R$, we have

$$|g(z)| = \left|z - \frac{z}{R} - \frac{\phi(z)u^{-1}}{R}\right| \leqq \left|z - \frac{z}{R}\right| + \frac{b_1 + b_2 + \cdots + b_r}{R}$$

$$\leqq R - 1 + \frac{R-2}{R} \leqq R.$$

Thus the mapping $z \to g(z)$ is a continuous transformation which carries the 4-dimensional hypersphere $|z| \leqq R$ into itself. By Brouwer's fixed point theorem [7] there exists a value z_0 such that $g(z_0) = z_0$, and by (7), (8), and (9) we have $f(z_0) = 0$.

References

1. B. H. Arnold, A topological proof of the fundamental theorem of algebra, this MONTHLY, vol. 56 (1949), pp. 465–466.

2. Samuel Eilenberg and Ivan Niven, The "fundamental theorem of algebra" for quaternions, Bulletin of the Amer. Math. Soc. vol. 50 (1944), pp. 246–248.

3. E. C. Titchmarsh, *The theory of functions*, Oxford (1932), p. 118.
4. A good bibliography of work up to 1940 is given in James A. Ward, A theory of analytic functions in linear associative algebras, Duke Math. Journal, vol. 7 (1940), pp. 233–248.
5. C. C. MacDuffee, *An introduction to abstract algebra*, John Wiley (1940), pp. 191–193.
6. B. L. van der Waerden, *Modern algebra I*, Frederick Ungar (1949), p. 227.
7. S. Lefschetz, *Introduction to topology*, Princeton (1949), p. 117.

SOLVABILITY OF QUARTICS BY MEANS OF SQUARE ROOTS

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1. Introduction. A geometric configuration A is constructible with ruler and compasses from a geometric configuration B if and only if the numbers defining A are obtainable from the numbers defining B by rational operations and extractions of real square roots.[1]

If $a_1, a_2, \cdots, a_m$ are given complex numbers, a number b is said to be obtainable from the a's by rational operations and extractions of square roots (not necessarily real) if there exists a (finite) sequence $b_1, b_2, \cdots, b_t$ such that $b_t = b$ and each b_i is either

(a) one of the a's, or

(b) the sum, difference, product, or quotient of b_j and b_k, where j and k are less than i (j and k not necessarily distinct), or

(c) a square root of some b_j, where $j < i$.

For brevity we shall then say that b is a *multiple square root* in the a's.

If b is a multiple square root in the a's, there may be many sequences of the kind described. In any one sequence there may be some values of i for which b_i is of more than one of the three types (a), (b), (c); but there will be a certain number of i's, say $p \geqq 0$, such that b_i is of type (c) and not of either of the types (a) or (b). If the smallest value of the p's corresponding to the different sequences is n, we shall say that b is a multiple square root in the a's of order n.

If $n = 0$, then b is said to be *rational* in the a's. In this case b is obtainable by a finite number of rational operations performed upon the a's and upon numbers obtainable from the a's by additions, subtractions, multiplications, and divisions.

It often happens that each of the numbers defining a geometric configuration A is a root of an algebraic equation whose coefficients are rational in the numbers defining configuration B. It is desirable, therefore, to have a criterion for determining when such an equation has roots which are multiple square roots in the numbers of B.

For a linear or quadratic equation there is no problem. For a cubic equation we have the following result.

THEOREM 1. *If p, q, r are rational in $a_1, a_2, \cdots, a_m$, then the equation $x^3 + px^2 + qx + r = 0$ has a root which is a multiple square root in the a's if and only if it has a root which is rational in the a's.*[2]

[1] L. E. Dickson, *First Course in the Theory of Equations*, 1922, p. 30.

[2] The proof is exactly like the proof of the theorem in Dickson, pp. 32–34.

It follows immediately from this theorem (or by direct means) that, with the same hypothesis, if one root of the cubic is a multiple square root in the a's, then all the roots are.

It is easily shown by means of this theorem that the trisection of certain angles, the duplication of the cube, and some other geometric constructions, are not possible with ruler and compasses.[3]

In this note we call attention to a simple criterion, similar to Theorem 1, for quartic equations.

2. Results. Let $f(x)=x^4+px^3+qx^2+rx+s$, where p, q, r, s are rational in $a_1, a_2, \cdots, a_m$. Let $g(y)=y^3-qy^2+(pr-4s)y+4qs-p^2s-r^2$ be the resolvent cubic of $f(x)$. We shall establish the following results.

THEOREM 2. *If $f(x)=0$ has a root which is a multiple square root in the a's, but which is not rational in the a's, then all its roots are multiple square roots in the a's.*

THEOREM 3. *All the roots of $f(x)=0$ are multiple square roots in the a's if and only if all the roots of $g(y)=0$ are multiple square roots in the a's.*

THEOREM 4. *The equation $f(x)=0$ has a root which is a multiple square root in the a's if and only if at least one of the two equations $f(x)=0$ and $g(y)=0$ has a root which is rational in the a's.*

As a simple application of Theorem 4, consider the problem of inscribing in a circle of radius 1 an isosceles triangle of area 1/4. If x is the square of one of the equal sides, we are led to the equation $x^4-4x^3+1=0$. The resolvent cubic equation is $y^3-4y-16=0$. Since neither of these equations has a rational root, the proposed construction is not possible with ruler and compasses.

Consider also the possibility of constructing 1/5 of an acute angle whose cosine is 61/64. If $x=\cos\theta$, where θ is the desired angle, then $\cos 5\theta=16x^5-20x^3+5x=61/64$. One root is 1/4, which is obviously not the desired solution. The reduced quartic equation is

$$x^4+\frac{1}{4}x^3-\frac{19}{16}x^2-\frac{19}{64}x+\frac{61}{256}=0,$$

which has

$$y^3+\frac{19}{16}y^2-\frac{263}{256}y-\frac{2529}{2048}=0$$

as its resolvent cubic. Neither of the last two equations has a rational root. Hence, the proposed construction is not possible with ruler and compasses.

3. Proof of Theorem 2. Let x_1 be a root of $f(x)=0$ which is a multiple square root in the a's of order $n>0$. Let $b_1, b_2, \cdots, b_t$ be a sequence of the kind

[3] Dickson, pp. 34–36.

described in §1 such that $b_i=x_1$, and such that for exactly n values of i, b_i is of type (c) and not of either of the types (a) or (b), and let b_l be the last such b_i. It follows easily that $x_1=\alpha+\beta\sqrt{k}$, where $\sqrt{k}$ is a square root of one of $b_1, b_2, \cdots, b_{l-1}$, and α and β are multiple square roots in the a's of order $n-1$ at most.

Proceeding as in Dickson *loc. cit.*, pp. 32–34, we find that $x_2=\alpha-\beta\sqrt{k}$ is also a root of $f(x)=0$, distinct from x_1. If x_3, x_4 are the other roots, then by eliminating x_3 from the equations

$$x_1+x_2+x_3+x_4=-p, \qquad x_1x_2+(x_1+x_2)(x_3+x_4)+x_3x_4=q,$$

we obtain

$$x_4^2+(p+x_1+x_2)x_4+q+(x_1+x_2)^2+p(x_1+x_2)-x_1x_2=0.$$

Since each of the coefficients of the powers of x_4 in this quadratic equation is a multiple square root in the a's, x_4 is also. It follows that the same is true of x_3.

4. Proof of Theorem 3. We recall[4] the following results.

(1) If y_i is a root of the resolvent cubic, then

$$f(x)=(x^2+\tfrac{1}{2}px+\tfrac{1}{2}y_i+\lambda_ix+\mu_i)(x^2+\tfrac{1}{2}px+\tfrac{1}{2}y_i-\lambda_ix-\mu_i),$$

where $\lambda_i^2=y_i+\frac{1}{4}p^2-q$, $2\lambda_i\mu_i=\frac{1}{2}py_i-r$, $\mu_i^2=\frac{1}{4}y_i^2-s$.

(2) *If* x_1, x_2, x_3, x_4 are the roots of the quartic, then

$$y_1=x_1x_2+x_3x_4, \qquad y_2=x_1x_3+x_2x_4, \qquad y_3=x_1x_4+x_2x_3$$

are the roots of the resolvent cubic.

Suppose x_1, x_2, x_3, x_4 are multiple square roots in the a's. From the expressions in (2) we see that y_1, y_2, y_3 are also.

Suppose, conversely, that one or more of y_1, y_2, y_3 is a multiple square root in the a's. From the expressions in (1) we see that the corresponding λ_i and μ_i are also. Since x_1, x_2, x_3, x_4 are the roots of the quadratic factors of $f(x)$, they also are multiple square roots in the a's.

5. Proof of Theorem 4. Suppose $f(x)=0$ has a root which is a multiple square root in the a's. If this root is rational in the a's, there is nothing more to be proved. If not, then by Theorem 2 all the roots of the quartic are multiple square roots in the a's. By Theorem 3 the same is true of the roots of $g(y)=0$. By Theorem 1, the cubic has a root which is rational in the a's.

Conversely, suppose $f(x)=0$ or $g(y)=0$ has a root which is rational in the a's. If $f(x)=0$ has such a root, there is nothing more to be proved. If $g(y)=0$ has such a root, then the last part of the proof of Theorem 3 applies and shows that all the roots of $f(x)=0$ are multiple square roots in the a's.

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[4] Dickson, *loc. cit.*, pp. 50–51.

CLASSROOM NOTES

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FROM SPHERICAL TRIGONOMETRY TO PLANE TRIGONOMETRY

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In the study of spherical trigonometry one is struck with the likeness of some of the spherical trigonometry formulas to certain plane trigonometry formulas. As for example, the law of sines in both trigonometries, the law of tangents for both, some of the right triangle formulas, and others show similarities. But in some cases there seems to be little or no similarity, as for example, the two laws of cosines. Also, there is a law of cosines for angles in spherical trigonometry while of course there is none like this or similar to it in plane trigonometry. And again, in what respect is the formula for area of a spherical triangle similar to or related to any one of the formulas for area of a plane triangle?

In this paper it is shown how, starting with the formulas in spherical trigonometry, to establish a correspondence between these and the formulas of plane trigonometry.

Consider the spherical triangle $\mathcal{ABC}$, Figure 1, and the plane triangle ABC, Figure 2. Letter the triangles in the usual manner and in Figure 1 let the center

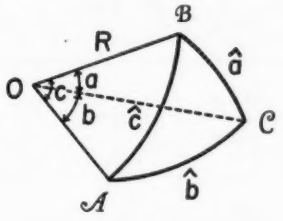

Fig. 1

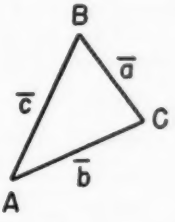

Fig. 2

of the sphere be O, and let the radius of the sphere be $R=O\mathcal{A}=O\mathcal{B}=O\mathcal{C}$. Also, let

a = angle $\mathcal{BOC}$ = arc $\mathcal{BC}$ in circular measure, Figure 1.

$\hat{a}$ = arc $\mathcal{BC}$ in linear measure $= Ra$, Figure 1;

$\bar{a}$ = side BC in linear measure, Figure 2;

and similarly for the b's and c's.

All angles will be measured in radians.

Next, hold the vertices of the triangle in Figure 1 fixed in space and let the center O of the sphere recede to infinity keeping $O\mathcal{A}=O\mathcal{B}=O\mathcal{C}$. Then $R\to\infty$, $\sphericalangle\mathcal{BOC}=a\to 0$, $\hat{a}\to\bar{a}$, $\sphericalangle\mathcal{A}\to\sphericalangle A$, and similarly for the remaining sides and angles. Also, area $\mathcal{K}$ of $\Delta\mathcal{ABC}\to$area K of ΔABC.

We shall make use of such simple limits as the limit of a sum, the limit of

a product, the limit of a quotient,

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1, \qquad \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1,$$

and

$$\lim_{\theta \to 0} \cos \theta = 1.$$

The equations will be *put in limit form* by replacing $\sin a$ by $Ra \cdot \sin a/a = \bar{a} \sin a/a$, the factor R being introduced either by multiplication of the equation by R or by multiplying the numerator and denominator of the term by R. Similarly for other sines and tangents of sides.

The formulas for the right triangles, plane and spherical, as given by Chauvenet, *A Treatise on Plane and Spherical Trigonometry*, tenth edition, 1908, page 108, for comparison are tabulated below, only here the order is changed and numbers are assigned.

	Spherical Right Triangle		Plane Right Triangle	
	a	b	c	d
1	$\sin \measuredangle A = \dfrac{\sin a}{\sin c}$	$\sin \measuredangle B = \dfrac{\sin b}{\sin c}$	$\sin A = \dfrac{\bar{a}}{\bar{c}}$	$\sin B = \dfrac{\bar{b}}{\bar{c}}$
2	$\cos \measuredangle A = \dfrac{\tan b}{\tan c}$	$\cos \measuredangle B = \dfrac{\tan a}{\tan c}$	$\cos A = \dfrac{\bar{b}}{\bar{c}}$	$\cos B = \dfrac{\bar{a}}{\bar{c}}$
3	$\tan \measuredangle A = \dfrac{\tan a}{\sin b}$	$\tan \measuredangle B = \dfrac{\tan b}{\sin a}$	$\tan A = \dfrac{\bar{a}}{\bar{b}}$	$\tan B = \dfrac{\bar{b}}{\bar{a}}$
4	$\sin \measuredangle A = \dfrac{\cos \measuredangle B}{\cos b}$	$\sin \measuredangle B = \dfrac{\cos \measuredangle A}{\cos a}$	$\sin A = \cos B$	$\sin B = \cos A$
5	$\cos c = \cos a \cdot \cos b$		$\bar{c}^2 = \bar{a}^2 + \bar{b}^2$	
6	$\cos c = \cot \measuredangle A \cdot \cot \measuredangle B$		$1 = \cot A \cdot \cot B$	

To show that 1a reduces to 1c put 1a in the limit form:

$$\sin \mathcal{A} = \frac{\sin a}{\sin c} = \frac{Ra \cdot \dfrac{\sin a}{a}}{Rc \cdot \dfrac{\sin c}{c}} = \frac{\hat{a} \cdot \dfrac{\sin a}{a}}{\hat{c} \cdot \dfrac{\sin c}{c}}.$$

On passing to limits we have 1c, that is,

$$\sin A = \frac{\bar{a}}{\bar{c}}.$$

In like manner formulas 1b, 2a, 2b, 3a, 3b transform respectively into 1d, 2c, 2d, 3c, 3d.

Formulas 4a, 4b, 6a give 4c, 4d, 6c, respectively, since $\cos a$, $\cos b$, and $\cos c$ become 1 in the limit. These formulas, 4c, 4d, and 6c imply that $A+B=\pi/2$, or $A+B+C=\pi$.

Formulas 5a and 5c merit special attention. On the surface it is difficult to see how they "correspond" except that they both involve the three sides. To establish this correspondence, square both sides of 5a, replace $\cos^2 a$ by $1-\sin^2 a$, and do likewise for $\cos^2 b$ and $\cos^2 c$. Then simplify and get

$$\sin^2 c = \sin^2 a + \sin^2 b - \sin^2 a \cdot \sin^2 b.$$

Put in limit form and get

$$\left(Rc \cdot \frac{\sin c}{c}\right)^2 = \left(Ra \cdot \frac{\sin a}{a}\right)^2 + \left(Rb \cdot \frac{\sin b}{b}\right)^2 - \left(Ra \cdot \frac{\sin a}{a}\right)^2 \cdot \sin^2 b,$$

or,

$$\hat{c}^2 \cdot \left(\frac{\sin c}{c}\right)^2 = \hat{a}^2 \cdot \left(\frac{\sin a}{a}\right)^2 + \hat{b}^2 \cdot \left(\frac{\sin b}{b}\right)^2 - \hat{a}^2 \cdot \left(\frac{\sin a}{a}\right)^2 \cdot \sin^2 b.$$

Passing to limits, we find 5c, that is

$$\bar{c}^2 = \bar{a}^2 + \bar{b}^2.$$

Formulas for the general triangle are treated in a similar manner, some transforming into the ordinary formulas of plane trigonometry and others into $A+B+C=\pi$. Three others will be exhibited.

The formula

$$\sin c \cdot \cos \mathcal{A} = \sin b \cdot \cos a - \cos b \cdot \sin a \cdot \cos \mathcal{C}, \tag{7}$$

see Chauvenet, (6), page 154, transforms into

$$\bar{b} = \bar{a} \cdot \cos c + \bar{c} \cdot \cos A,$$

that is, side $\bar{b}$ of the plane triangle equals the sum of the projections of sides $\bar{a}$ and $\bar{c}$ on $\bar{b}$.

Now consider the formula

$$\mathcal{K} = R^2E, \tag{8}$$

for the area of the spherical triangle $\mathcal{ABC}$, where $E=\mathcal{A}+\mathcal{B}+\mathcal{C}-\pi$ is the spherical excess.

If we use Lhuillier's formula (Chauvenet (228), page 230) for the spherical excess we can write (8) as follows:

$$\begin{aligned}\mathcal{K} &= 4R^2\cdot\frac{\frac{1}{4}E}{\tan\frac{1}{4}E}\cdot\left[\tan\tfrac{1}{2}s\cdot\tan\tfrac{1}{2}(s-a)\cdot\tan\tfrac{1}{2}(s-b)\cdot\tan\tfrac{1}{2}(s-c)\right]^{1/2}\\ &= \frac{\frac{1}{4}E}{\tan\frac{1}{4}E}\cdot\left[\left(Rs\cdot\frac{\tan\frac{1}{2}s}{\frac{1}{2}s}\right)\cdot\left(R(s-a)\cdot\frac{\tan\frac{1}{2}(s-a)}{\frac{1}{2}(s-a)}\right)\cdot\cdots\right]^{1/2}\\ &= \frac{\frac{1}{4}E}{\tan\frac{1}{4}E}\cdot\left[\mathfrak{s}\cdot\frac{\tan\frac{1}{2}s}{\frac{1}{2}s}\cdot(\mathfrak{s}-\mathfrak{a})\cdot\frac{\tan\frac{1}{2}(s-a)}{\frac{1}{2}(s-a)}\cdot\cdots\right]^{1/2},\end{aligned}$$

where $s=\frac{1}{2}(a+b+c)$. On passing to limits, we find the formula for the area of the plane triangle ABC in terms of the sides of the triangle, with $\mathfrak{s}=\frac{1}{2}(\mathfrak{a}+\mathfrak{b}+\mathfrak{c})$.

If instead of Lhuillier's formula for the spherical excess, we use formula (230), page 231, of Chauvenet's trigonometry, the area can be written

$$\begin{aligned}\mathcal{K} = R^2E &= R^2\cdot\frac{E}{\sin\frac{1}{2}E}\cdot\frac{\sin\frac{1}{2}a\cdot\sin\frac{1}{2}b}{\cos\frac{1}{2}c}\cdot\sin\mathcal{C}\\ &= 2\,\frac{\frac{1}{2}E}{\sin\frac{1}{2}E}\cdot\frac{Ra}{2}\cdot\frac{\sin\frac{1}{2}a}{\frac{1}{2}a}\cdot\frac{Rb}{2}\cdot\frac{\sin\frac{1}{2}b}{\frac{1}{2}b}\cdot\frac{\sin\mathcal{C}}{\cos\frac{1}{2}c}\\ &= \frac{1}{2}\cdot\frac{\frac{1}{2}E}{\sin\frac{1}{2}E}\cdot\mathfrak{a}\mathfrak{b}\cdot\frac{\sin\frac{1}{2}a}{\frac{1}{2}a}\cdot\frac{\sin\frac{1}{2}b}{\frac{1}{2}b}\cdot\frac{\sin\mathcal{C}}{\cos\frac{1}{2}c}.\end{aligned}$$

On passing to limits, we get

$$K = \tfrac{1}{2}ab\cdot\sin C,$$

the formula giving the area of the plane triangle ABC in terms of two sides and the included angle.

SEMINAR PROGRAM IN MATHEMATICS AT REGIS COLLEGE

SISTER MARY LEONARDA, Regis College

The seminar program was inaugurated at Regis College in the fall of 1947 in an effort to provide for each concentrator a testing ground where she would meet the problems of her field in a setting reproducing as nearly as possible the demands upon her knowledge and training and use of newly acquired knowledge which she would encounter after graduation. Although still in the experimental stage, the program has been undeniably effective. In the present paper, the writer gives some aspects of the seminar system as it has been developed thus far in the Department of Mathematics. The topics for the current year are listed

as typical of the content of the discussions but do not indicate permanency. Classroom experiences and current demands determine the details and contribute significantly to the success of a seminar program by keeping it from becoming too highly systematized and nonadjustable.

Our junior concentrators meet once each week for a two-hour seminar which treats of geometry. The junior seminar is a reading seminar which is made possible because the student comes to it reasonably prepared with knowledge and skills gained from her lower level courses in mathematics. It is directed to stimulate critical and systematic reading of monographs, source material, books of reference, biography, popular writings. Occasionally a text is reviewed. In general, texts are not included since their content is already a digest of some greater finding, and therefore provides no experience of the type we seek. Since "understanding of mathematics cannot be transmitted by painless entertainment any more than education in music can be brought by the most brilliant journalism to those who have never listened intensively,"* an important supplement to our reading requirement is the actual performance of a fair number of proposed problems and "actual contact with the content of living mathematics."* The author quoted points out that such contact is rendered possible at this level if advanced technicalities and profound methods can be laid aside for the purpose of disclosing the goal.

The following topics indicate the trend of our interests in the current reading seminar:

First Quarter: Fundamental Constructions; Constructible Numbers; Three Greek Problems; I Methods of Construction—Inversion; II Methods of Construction—Mascheroni Constructions; III Methods of Construction—Straightedge, Carpenter's Square; Peaucellier Cells; Problem of Apollonius; Reflection in a System of Three Circles.

Second Quarter: Pascal's Theorem; Brianchon's Configuration; Polar Coordinates; Cylindrical Coordinates; Spherical Coordinates; Homogeneous Coordinates; Transformations; The Straight Line.

Third Quarter: Conic Sections Generated by Projective Pencils; Linkages; Cycloids, Tractrix, Catenary; Surfaces of Revolution; Hyperboloid of One Sheet and Hyperbolic Paraboloid; The Pseudosphere; Riemann.

Fourth Quarter: Klein's Non-Euclidean Model; Poincaré's Non-Euclidean Model; Euler's Formula for Polyhedra; The Four-Color Problem; Map Projection; One-sided Surfaces; What is Mathematics?

The required reading pertinent to this outline is taken from the fields of foundation and development of mathematics, algebra, and geometry. It includes the works of such writers as Young, Hardy, Heath, Smith, Baker, Frost, Salmon, Klein, Hudson, Kasner, Thomas, Albert; source material on the topics treated briefly in D. E. Smith's *Source Book in Mathematics*,† and others such as Peacock's *Treatise on Algebra*,‡ and studies on the Rhind papyrus, are expected to arouse further investigation into primary source material.

* R. Courant and H. Robbins: *What is Mathematics?*, Oxford University Press, New York, 1943, p. v.

† McGraw-Hill Book Company, New York, 1929.

‡ Scripta Mathematica Press, New York, 1940.

Not the least important function of the junior seminar is the pursuit of a chosen subject which, for want of a better name, we call a hobby. This hobby is acknowledged and encouraged in our seminar system. The studies pursued are varied: architecture, clocks, market research, inversion, bridges, airplanes, Mascheroni constructions, singular solutions of differential equations, the history of the methods used in assigning the number of representatives in Congress for a given area. Some of the studies have been written formally and bound and placed in our departmental library. A part of the study on airplanes was published, and we are hoping to have the study of bridges appear. With few exceptions the experience which has been gained in the pursuit of the hobby remains only the joy of personal achievement.

Our senior concentrators participate in a coordinating seminar on analysis. The seminar on geometry provides a magnificent foundation upon which to build the coordinating seminar. The following discussions constitute our program for the present coordinating seminar:

First Quarter: The Rhind Papyrus; Pythagorean Arithmetic; Irrational Numbers; Complex Numbers; DeMoivre's Formula and the Roots of Unity; The Number Pi; Euler's Number e.

Second Quarter: Calculating Devices; The Sand-Reckoner and His Successors; The Fundamental Theorem of Algebra and Algebraic Equations; Trigonometric and Transcendental Equations; Algebra of Sets—Application to Mathematical Logic.

Third Quarter: Variables, Functions, Graphs; Limits by Continuous Approach; Abel on the Continuity of Functions Defined by Power Series; Theorems of Bolzano and Weierstrass; Heron's Theorem; Extremum Property of Light Rays; Tangent Properties of Ellipse and Hyperbola; Corresponding Extremum Properties; Schwartz' Triangle Problem; The Cycloid.

Fourth Quarter: Maxima and Minima of Functions of Several Variables; Saddle Points; Soap Film Experiments; Plateau's Problem; Pre-Calculus Integration as a Limiting Process; Integral Calculus; Fermat, Newton, Leibniz and the Derivative; Geometric Meanings of the First and Second Derivatives; Differential Equation of the Exponential Function; Radioactive Disintegration; Law of Growth; Compound Interest.

The works of Archimedes, Cauchy, Abel, Dedekind, Hardy, Courant, Ferrar, Young, Whittaker and Watson, Birkhoff and Maclane, N. Bourbaki, Scarborough, and many others which are of significance in the investigations of the students, should provide dependable and interesting reading matter with which to develop the proposed topics. *What is Mathematics*? by R. Courant and H. Robbins contains an excellent directive reading on each topic.

Each seminar course puts the instructor on his mettle as well as the student. The method is an attempt to vitalize instruction and to give a complete view of the field such as cannot be obtained in isolated courses. It puts the burden of his own education on the student where it belongs and is adaptable to his powers. Neither student nor instructor can avoid work. Yet many students and instructors alike do not know how we taught without it, or even how we managed with the lesser forms of seminar which were the forerunners of our present system.

FINDING THE PARTICULAR INTEGRAL IN A SPECIAL CASE

F. A. FICKEN, University of Tennessee and New York University

In a first course in differential equations, considerable attention is usually devoted to the linear equation $f(D)y=g(x)$, where $f(D)$ is a polynomial in D with constant coefficients. If the equation is non-homogeneous, one can find a particular integral in any case by variation of parameters, but special methods are often used to take advantage of the specific character of the right member of the equation. In certain cases, including some of practical interest, the right member is of the form $P(x)e^{\mu x}$, where $P(x)$ is a polynomial and μ is a constant. The object of this note is to emphasize the value of a particular combination of standard processes for handling this specific problem.

We assume throughout that x is a real variable and that the right member $g(x)$ is a linear combination of terms of the form $x^k e^{ax}\cos bx$ and $x^k e^{ax}\sin bx$ with k a non-negative integer and a and b real. The complementary function is not used.

The steps are as follows:

1. Express sines and cosines as imaginary exponentials. Then $g(x)$ is a linear combination of terms of the form $x^k e^{\mu x}$ with μ complex. Next, collect terms with the same μ. Then $g(x)$ is a sum of terms of the form $P(x)e^{\mu x}$, $P(x)$ being a polynomial. For each such term separately, use steps 2–5 to obtain a partial particular integral y_μ, and finally add the y_μ to obtain the whole particular integral.

Note that, if the coefficients of f are real, then y_μ for the complex conjugate of $P(x)e^{\mu x}$ will be the complex conjugate of y_μ for $P(x)e^{\mu x}$, and y_μ for the real (imaginary) part of $P(x)e^{\mu x}$ will be the real (imaginary) part of y_μ for $P(x)e^{\mu x}$. If, as often happens, the coefficients of $P(x)$ are real, one finds y_μ and uses its real or imaginary part according as $e^{\mu x}$ arose from a cosine or a sine. This common technique is illustrated in the example below.

2. Let $y=ze^{\mu x}$ and cancel $e^{\mu x}$, getting $f(D+\mu)z\equiv Q(D)z=P(x)$, where $Q(D)$ is a polynomial whose coefficients are generally complex if μ is.

3. Let $D^r z=w$, choosing r so that the constant coefficient of the polynomial operating on w in the left member is not zero.

4. The equation then has the form $Q_n(D)w=P_m(x)$, where $P_m(x)$ and $Q_n(D)$ are polynomials of degree indicated by the subscripts, and the "constant term" of $Q_n(D)$ *does not vanish*. Using undetermined coefficients, find a polynomial $w_m(x)$ *of degree* m that satisfies this equation. (See lemma at end of note.)

5. Obtain $z_{m+r}(x)$ by integrating $w_m(x)$ r times, letting constants of integration vanish. The required partial particular integral is then $y_\mu=z_{m+r}(x)e^{\mu x}$ (verification trivial).

The process may be illustrated by the following example:

$$f(D)y \equiv (D^3 - 5D^2 + 17D - 13)y \equiv (D - 1)(D^2 - 4D + 13)y$$

$$= \left(\frac{1}{10} - 6x\right) e^{2x}\cos 3x.$$

1. $(D^3-5D^2+17D-13)\,Y=(1/10-6x)e^{(2+3i)x}$. Since the coefficients of f and P are real, y is the real part of Y.

2. Let $Y=ze^{(2+3i)x}$, getting $f(D+2+3i)z\equiv[D^3+(1+9i)D^2-6(3-i)D]z=1/10-6x$.

3. With $w=Dz$, $[D^2+(1+9i)D-6(3-i)]w=1/10-6x$.

4. Substitute $w=\alpha+\beta x$ and equate coefficients, arriving at

$$w=-\frac{7(7-11i)}{6\cdot 10^2}+\frac{3+i}{10}x=Dz.$$

5. $$z=\int w\,dx=-\frac{7(7-11i)}{6\cdot 10^2}x+\frac{3+i}{2\cdot 10}x^2=Ye^{-(2+3i)x}$$

$$y=Rl(Y)=\frac{1}{6\cdot 10^2}[(-49+90x)\cos 3x-(77+30x)\sin 3x]xe^{2x}.$$

The factual information needed by the student is as follows: Step 1 and a possible final step (as from Y to y in the example) assume familiarity with the algebra of real and complex numbers and with the basic fact that $e^{it}=\cos t+i\sin t$. Step 2 merely "eliminates the exponential"; this is tactically advantageous in many similar situations. Step 3 introduces a new dependent variable so that the differential equation shall be of lowest order consistent with its form and our ignorance of the complementary function; one does this anyhow. Step 4 makes as economical a use as possible of undetermined coefficients. The particular choice of a trial function is governed by a simple *general* rule requiring no operations on the right members and no knowledge of whether or not or to what order $f(\mu)$ may vanish. Step 5 merely compensates for steps 3 and 2.

This factual equipment is quite modest. It is general and basic rather than specific and *ad hoc*. Most of it is used in earlier parts of the course, and the entire process may be illustrated—perhaps informally—in connection with the *first* order linear equation with constant coefficients. Students with this previous experience accept quite readily the combination of several simple, natural, familiar steps into a method or process, and it quickly becomes automatic. They welcome the absence of formulas and the economy and effectiveness of the process. The algebraic work seems often to be less massive than that required by other processes.

The effectiveness of step 4 is guaranteed by the following well known lemma whose proof is easy.

LEMMA: *Let m and n be non-negative integers. Let $q_0, \cdots, q_n$ be complex constants with $q_0\neq 0$ and $q_n\neq 0$, and let $Q_n(D)\equiv\sum_{j=0}^{n}q_jD^j$, with $D\equiv d/dx$. Let $p_0, \cdots, p_m$ be complex constants with $p_m\neq 0$, and let $P_m(x)\equiv\sum_{k=0}^{m}p_kx^k$.*

Then unique complex constants $a_0, \cdots, a_m$ exist, with $a_m\neq 0$, such that $y_m(x)=\sum_{k=0}^{m}a_kx^k$ satisfy the differential equation $Q_n(D)y=P_m(x)$. The a_k are real if all the p_k and q_j are real.

Students quickly learn, if $n>m$, to discard those q_j for which $j>m$. For example, if $m=0$, $Q_n(D)y=p_0$ has the solution $a_0=p_0/q_0$.

It may be mentioned that a very limited search has revealed a similar discussion of the special problem treated here only in Goursat's *Cours d'Analyse* (vol. II, pp. 458 ff. in fifth edition).

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 911. *Proposed by G. B. Huff, University of Georgia*

The following item appeared in the April 1, 1913, issue of the Erewhon Daily Howler: "Science lovers of Guayazuela will be disappointed to learn that the antarctic expedition of Professor Euclide Paracelso Bombasto Umbugio, which was planned to start the first of next month and last through November, has been temporarily postponed. In making his characteristically thorough preparations, the eminent numerologist discovered that June 13 of this year falls on Friday and has decided that he will postpone the trip until a year when no Friday the thirteenth occurs in the seven months of the expedition. The Professor, an authority on the doctrine of probability, assured the press that the delay will be of short duration. He explained that the probability of an auspicious month is 6/7 and that hence the probability of seven consecutive auspicious months is $(6/7)^7$ or a little more than 2/5. From this the genius infers that the probability of postponing the trip next year is less than 3/5 and the probability that it will be postponed for nine consecutive years is $(3/5)^9$ or less than 1/100."

Of course, we now know that the Professor has never made his trip, though four times nine years have passed. Is the Professor having a tremendous run of bad luck?

E 912. *Proposed by H. E. G. P.*

Show that the infinite product

$$\prod_{n=1}^{\infty} (n+1)^3/n(n+x)(n+2x)$$

represents, in all points of its set of convergence, the rational function $2x$. (By permission of Prof. E. P. B. Umbugio, April 1, 1950.)

E 913. *Proposed by C. S. Ogilvy, Columbia University*

An isosceles triangle ABC has vertex angle $C=20°$. Points M and N are taken on AC and BC such that angle $ABM=60°$ and angle $BAN=50°$. Prove that angle $BMN=30°$. (Attention Prof. E. P. B. Umbugio.)

E 914. *Proposed by C. W. Trigg, Los Angeles City College*

It is desired to arrange congruent equilateral triangles, with at least one side of each triangle coinciding with a side of another triangle, in a plane configuration which can be folded along the common sides into a tetrahedron with no open edges. (An open edge is one through which there is straight line access to the centroid of the solid figure.)

(1) What is the smallest number of triangles necessary?

(2) How many distinct configurations containing this smallest number may be folded into a closed tetrahedron?

E 915. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The circle passing through the feet L, M, N of the cevians AL, BM, CN which divide the sides $BC=a$, $CA=b$, $AB=c$ of a triangle in the ratios

$$BL/LC = c^n/b^n, \qquad CM/MA = a^n/c^n, \qquad AN/NB = b^n/a^n,$$

intercepts on the lines BC, CA, AB signed segments of lengths x, y, z satisfying the relation

$$x/a^{n-1} + y/b^{n-1} + z/c^{n-1} = 0.$$

SOLUTIONS

Three Arithmetic Progressions

E 877 [1949, 473]. *Proposed by L. J. Burton, Bryn Mawr College*

Each of three arithmetic progressions continues indefinitely in both directions, and each has a difference which is an integer. Prove that if there is a term common to each pair of progressions then there is a term common to all three progressions.

Solution by J. H. Butchart, Arizona State College at Flagstaff. Without loss of generality we may increase or decrease all numbers so that the common term of progressions A and B is zero, and we may change the unit so that the common differences a, b, c will have no divisor greater than 1 common to all three. Thus A consists of multiples of a and B of multiples of b. Let jb be the term common to B and C and let $jb+kc$ be the term common to C and A, or otherwise ia, showing that k contains the greatest common divisor of b and a. Consider the term $jb+kc+pac$ common to A and C. It will be a term of B if

$k+pa=qb$, where p and q are to be determined. By Euclid's algorithm p and q may be found so that $qb-pa=k$, thus completing the proof.

Also solve by Max Beberman, D. H. Browne, Aaron Buchman, W. F. Cahill, M. S. Klamkin, Roger Lessard, T. F. Mulcrone, S. T. Parker, Azriel Rosenfeld, R. W. Shoemaker, and the proposer.

Relations in a Right Triangle

E 878 [1949, 473]. *Proposed by Kaidy Tan, Anglo-Chinese College, Amoy, China*

Let S be the incenter of the right triangle ABC, and X the point of contact of the hypotenuse BC with the incircle. With center X and radius XS describe the circle cutting BS, CS at M and N respectively. Let AD be the altitude on BC. Show that M, N are the incenters of the right triangles ABD and ACD respectively.

Solution by C. S. Ogilvy, Columbia University. Describe the circle with radius XS and center at Z, the point of contact of AB with the original incircle. It is clear that this circle also cuts SB at M. Let it cut ZB at Q. Then $\sphericalangle BAD = 90° - \sphericalangle ABD = 90° - 2\sphericalangle ABM = 90° - (90° - \overset{\frown}{MQ}) = \overset{\frown}{MQ} = 2\sphericalangle QAM$. That is, M lies on the bisector of $\sphericalangle BAD$, and is therefore the incenter of triangle BAD. Similarly it can be shown that N is the incenter of triangle CAD.

Also solved by Leslie Booth, Aaron Buchman, W. E. Buker, J. H. Butchart, P. W. M. John, L. M. Kelly, Sam Kravitz, Joseph Langr, Roger Lessard, F. L. Miksa, M. C. Stapp, C. W. Trigg, Roscoe Woods, and the proposer.

Trigg gave three solutions and remarked that other properties of the figure of the problem can be found in the article, *Relating to some relations in a right-angled triangle*, by Albert Babbitt, this MONTHLY [1918, 347–8].

Minimal Path Connecting n Points

E 880 [1949, 474]. *Proposed by Peter Ungar, University College, London, England*

Let n points be given in the plane, not all on a straight line. The shortest closed route connecting them is a simple polygon.

Solution by O. D. Smith, Oregon State College. Connect the n points by any closed path. Obviously a shorter path is obtained by connecting the points in the same order by a polygonal line with the given points as the only vertices. Suppose that segments P_iP_{i+1} and P_kP_{k+1} of this polygonal path intersect in a point P, so that the path followed is $P_iPP_{i+1} \cdots P_kPP_{k+1} \cdots P_i$. If P is not a point of the given set, then $P_iP_k \cdots P_{i+1}P_{k+1} \cdots P_i$ is a shorter polygonal path and does not contain the intersection P. If P is a point of the given set, then $P_iPP_k \cdots P_{i+1}P_{k+1} \cdots P_i$ is a shorter polygonal path and does not contain P as an intersection. This proves the theorem.

Also solved by L. M. Kelly, D. W. Matlack, E. A. Nordhaus, and the proposer

Factorial Series

E 881 [1949, 552]. *Proposed by C. M. Sandwick, Easton High School, Easton, Pa.*

(a) Show that any positive integer can be represented uniquely in the form $\sum_{n=1}^{\infty} a_n n!$, $0 \leqq a_n \leqq n$.

(b) Show that any positive rational number less than unity can be represented uniquely in the form $\sum_{n=1}^{\infty} a_n/(n+1)!$, $0 \leqq a_n \leqq n$.

Solution by Aaron Buchman, Buffalo, N.Y. (a) Let k be any positive integer and choose m such that $m! \leqq k < (m+1)!$. Then we have

$$k = a_m m! + r_m, \qquad 0 \leqq a_m \leqq m, \qquad 0 \leqq r_m < m!. \tag{1}$$

Again,

$$r_m = a_{m-1}(m-1)! + r_{m-1}, \qquad 0 \leqq a_{m-1} \leqq m-1, \qquad 0 \leqq r_{m-1} < (m-1)!.$$

Proceeding in this manner it is evident that

$$k = \sum_{n=1}^{\infty} a_n n!, \qquad 0 \leqq a_n \leqq n, \tag{2}$$

and representation of k in the required form is always possible. It is evident that $a_n = 0$ when $n > m$.

Since $h! = (h-1)(h-1)! + (h-1)!$, it follows that

$$h! = 1 + \sum_{n=2}^{h-1} (n-1)(n-1)!. \tag{3}$$

Now if, in (1), a_n should be decreased, the new remainder would no longer be less than $m!$. Then, by (3), it would be impossible to complete the summation and keep the rest of the a_i within the limits $0 \leqq a_i \leqq i$. Thus a_m may not be decreased, and of course it cannot be increased. Similarly $a_{m-1}, a_{m-2}, \cdots$ may not be altered, and the representation (2) is unique.

(b) Let u/v be any positive rational number, in lowest terms, with $u < v$. Then we have

$$u/v = a_1/2! + r_1, \qquad 0 \leqq a_1 \leqq 1, \qquad 0 \leqq r_1 < 1/2!. \tag{4}$$

Again,

$$r_1 = a_2/3! + r_2, \qquad 0 \leqq a_2 \leqq 2, \qquad 0 \leqq r_2 < 1/3!.$$

Proceeding in this manner it is evident that

$$u/v = \sum_{n=1}^{\infty} a_n/(n+1)!, \qquad 0 \leqq a_n \leqq n, \tag{5}$$

and representation of u/v in the required form is always possible. Let $(m+1)!$

be the least factorial which is divisible by v. Then, since

$$r_m = u/v - \sum_{n=1}^{m} a_n/(n+1)!, \qquad 0 \leqq r_m < 1/(m+1)!,$$

we see that $r_m = 0$, and consequently $a_n = 0$ when $n \geqq m+1$.

Since $1/h! = h/(h+1)! + 1/(h+1)!$, it follows that

$$1/h! = \sum_{n=h}^{\infty} n/(n+1)!. \tag{6}$$

Now if, in (4), $r_1 \neq 0$ and a_1 should be decreased, the new remainder would exceed $1/2!$. Then, by (6), it would be impossible to complete the summation and keep the rest of the a_i within the limits $0 \leqq a_i \leqq i$. Thus, if $r_1 \neq 0$, a_1 may not be decreased, and of course it cannot be increased. Similarly $a_2, a_3, \cdots, a_{m-1}$ may not be altered. It then follows, from (5) and (6), that only a_m may be decreased by one, and the following zero numerators replaced by $m+1, m+2, \cdots$. Thus there are *two* ways of representing u/v in the required form; one is a unique finite development, and the other is a unique infinite development.

Also solved by M. S. Klamkin, N. D. Lane, Roger Lessard, G. M. Merriman, C. S. Ogilvy, and the proposer. Some of these solutions only showed the possibility of representation in the required form but failed to deal adequately with the question of uniqueness.

Merriman pointed out that part (b) appears, in slightly sharper form, as ex. 12, p. 49, Birkhoff-MacLane, *A Survey of Modern Algebra*, and as ex. 2, p. 31, Hardy, *Pure Mathematics*, 6th ed. Arthur Rosenthal mentioned that although there are exactly two ways of representing a positive rational number less than unity in the required form, any irrational number less than unity has a unique representation. He said that these results were obtained explicitly by M. Stéphanos, *Bull. Soc. Math. France*, vol. 7 (1879), pp. 81–83, but are contained in even older and more general discussions of G. Cantor, *Zeitschrift für Math. u. Physik*, vol. 14 (1869), pp. 121–128. Series of this sort were later called *Cantor series*. See, *e.g.*, O. Perron, *Irrationalzahlen*, Berlin (1st ed., 1921; 2nd ed., 1939; 3rd ed., 1947), sec. 33. For the situation where the series has periodic coefficients see V. E. Dietrich and A. Rosenthal, *Bull. Amer. Math. Soc.*, vol 55 (1949), pp. 594–596.

The proposer stated the following generalization of part (a): *If $f(n)$ is a positive integer when n is a non-negative integer, and if $f(0) = 1$, any non-negative integer can be represented uniquely in the form*

$$\sum_{n=1}^{\infty} a_n \prod_{m=0}^{n-1} f(m), \qquad 0 \leqq a_n < f(n).$$

Thus, when $f(n) = n+1$, we have $N = \sum a_n n!$, $0 \leqq a_n < n+1$; when $f(n) = (n+1)^k$, we have $N = \sum a_n (n!)^k$, $0 \leqq a_n < (n+1)^k$; when $f(n) = k^n$ we have $N = \sum a_n k^{n(n-1)/2}$,

$0 \leqq a_n < k^n$; *etc.* Representation of a number in any scale of notation is also a special case. Let $f(n) = b$ when $n > 0$ and let $f(0) = 1$, then $N = \sum a_n b^{n-1}$, $0 \leqq a_n < b$.

Five Tetrahedra Forming a Decahedron

E 882 [1949, 552]. *Proposed by C. W. Trigg, Los Angeles City College*

Five regular tetrahedra arranged around a common edge just fail to fill space around that edge. (a) What is the closest approximation to a regular tetrahedron such that five such tetrahedra will fill the space around a common edge to form a decahedron having equilateral faces? (b) Show that the edge of the decahedron is $\sqrt{5}$ times the radius of the sphere touching those edges which radiate from the axis of the decahedron.

Solution by C. M. Sandwick, Easton High School, Easton, Pa. (a) Let *P-QAB* be one of the tetrahedra, forming with the other tetrahedra the decahedron *P-ABCDE-Q*. The plane of the regular pentagon *ABCDE* bisects *PQ* perpendicularly at *O*. Draw *OA*. If the edge *AB* is taken as unity, $OA = \sqrt{(5+\sqrt{5})/10}$. But $PA = 1$. Therefore $PO = \sqrt{(5-\sqrt{5})/10}$. Then $PQ = 2PO \doteq 1.05$. The distortion from regularity consists of an increase of slightly more than 5% in the length of *PQ*.

(b) Draw *OR* perpendicular to *PA*. Then $OR/OA = PO/PA$, or $OR = 1/\sqrt{5}$. Hence $PA = OR\sqrt{5}$.

Also solved by Roger Lessard and the proposer.

***N*-ics Having *n*-1 Roots in a Given Interval**

E 884 [1949, 552]. *Proposed by E. P. Starke, Rutgers University*

Show that there exist infinitely many rational integral equations with integral coefficients and leading coefficient unity, and of degree n, such that $n-1$ of the roots occur within a specified interval, however small.

Solution by Paul Erdös, Syracuse University. Let $x_1, x_2, \cdots, x_{n-1}$ be $n-1$ arbitrary distinct rational numbers in the specified interval and consider

$$x^n \pm c_n \prod_{i=1}^{n-1} (x - x_i) = 0, \tag{1}$$

where c_n is a large integer so chosen that all coefficients of (1) are integers. Now if c_n is sufficiently large then equation (1) has $n-1$ roots as close as we please to $x_1, x_2, \cdots, x_{n-1}$, and all conditions are met.

Also solved by C. S. Ogilvy and the proposer.

It is another matter to find the n-ic with smallest coefficients. The proposer gave $x^5+22x^4-47x^3+34x^2-10x-1$ as a quintic with four distinct roots between 0 and 1, and $x^4+12x^3-22x^2+12x-2$ as a quartic with three distinct roots between 0 and 1.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTIONS

4390. *Proposed by Roshan Lal Gupta, Government College, Ludhiana, India*

If $[x]$ denotes the greatest integer function, prove that

$$\sum_{q\geq 1} [(x/q^m)^{1/n}] = \sum_{q\geq 1} [(x/q^n)^{1/m}].$$

Is there any relation between the terms of the two sums?

4391. *Proposed by P. T. Bateman, Institute for Advanced Study*

Given a fixed positive integer k and a complex-valued function $f(n)$ defined on the positive integers and such that $f(n_1)=f(n_2)$ for $n_1\equiv n_2 \pmod{k}$, $|f(n)|\leq 1$ for all n, $f(n)=0$ for $(n, k)>1$, and $\sum_{n=1}^{k} f(n)=0$. Show that

$$\left|\sum_{k=1}^{\infty} \frac{f(n)}{n}\right| < \log k.$$

4392. *Proposed by Paul Erdös, University of Illinois*

Let $f(z)$ be analytic for $|z|\leq 1$. Let z_0 be the point $(|z_0|=1)$ where $f(z)$ assumes its maximum on the unit circle. Prove that $f'(z_0)\neq 0$.

4393. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that: (1) The center ω of an equilateral hyperbola $(\mathfrak{H})$ inscribed in a triangle ABC, the orthocenter H of this triangle, and the center N of the nine-point circle (N) of the triangle of contact $\alpha\beta\gamma$ of $(\mathfrak{H})$ with the sides of ABC are collinear. (2) The polar circle of triangle ABC is tangent to the circle (N). (3) If BC, CA, AB; $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ respectively intersect one of the asymptotes of $(\mathfrak{H})$ in A', B', C'; α', β', γ', we have

$$\omega N/\omega H = \omega\alpha'\cdot\omega\beta'\cdot\omega\gamma'/\omega A'\cdot\omega B'\cdot\omega C'.$$

4394. *Proposed by H. F. Sandham, Dublin, Ireland*

Evaluate

$$\int_0^{\infty} \frac{\log x}{e^x+1}\,dx.$$

SOLUTIONS

Mutually Inscribed Tetrahedra

4279 [1948, 34]. Corrected. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Prove that a tetrahedron $ABCD$ whose vertices A, B, C are the inverses, with respect to a sphere with center D, of a right-angled triangle, and the tangential tetrahedron $A_1B_1C_1D_1$ of that tetrahedron are each inscribed in the other.

Solution by the Proposer.* 1. Preliminaries. If the vertices of a triangle ABC, of sides a, b, c, are transformed by an inversion of center D, with D exterior to the plane ABC, and power k, so that the points A', B', C' thus obtained are the vertices of a right triangle $A'B'C'$, with $\overline{B'C'}^2=\overline{C'A'}^2+\overline{A'B'}^2$, we have $B'C'=ka/DB\cdot DC$, $C'A'=kb/DC\cdot DA$, $A'B'=kc/DA\cdot DB$, whence we obtain

$$-a^2\cdot\overline{DA}^2+b^2\overline{DB}^2+c^2\overline{DC}^2=0. \tag{1}$$

The locus of points D satisfying (1) is a sphere whose center is the pole A_1 of BC with respect to the circumcircle of triangle ABC. The barycentric coordinates of A_1 with respect to the reference triangle ABC are $-a^2$, b^2, c^2.

2. By (1) the edges $BC=a$, $AD=a'$, $CA=b$, $DB=b'$, $AB=c$, $DC=c'$, of the tetrahedron $T\equiv ABCD$ satisfy

$$(aa')^2=(bb')^2+(cc')^2.$$

LEMMA. *The antiparallel sections in each of the trihedral angles of the tetrahedron T are right triangles for which the vertices of the right angles are on the edges BC and DA.* The antiparallel sections are similar to the associated triangle of tetrahedron T, since their sides are proportional to aa', bb', cc'. The vertices of the right angles of antiparallel sections in the trihedral angles (B) and (C) are on the edge BC and those of antiparallel sections in the trihedral angles (D) and (A) are on the edge DA.

THEOREM. *The faces CDA and DAB are conjugate with respect to the circumsphere of T. The same statement holds for the faces ABC and BCD.* The plane (π) tangent at A to the circumsphere (O) of T cuts the planes of the faces CDA and DAB in two perpendicular lines Δ_a and Δ_a' (Lemma). The line B_1A which joins A to the pole B_1 of face CDA with respect to (O) is in plane (π) where it is perpendicular to the tangent at A of the circumcircle of triangle CDA: as this tangent line is in the plane (π) it must coincide with the line Δ_a. Hence the line B_1A is the line Δ_a'. Thus the pole B_1 of plane CDA is in plane DAB and the pole C_1 of the latter is in plane CDA. Likewise the pole D_1 of plane ABC is in plane BCD and the pole A_1 of the latter is in plane ABC.

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* Translated and checked by W. E. Byrne, Virginia Military Institute.

COROLLARY. *The tetrahedron T and its tangential tetrahedron $T_1 \equiv A_1B_1C_1D_1$ are inscribed one in the other.*

Tetrahedron and Inscribed Sphere

4298 [1948, 320]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$, the incenter of which is I, the perpendiculars at A to the faces ACD, ADB, ABC respectively cut the planes ICD, IDB, IBC in A_1, A_2, A_3. (1) The perpendicular through A to the plane $A_1A_2A_3$ passes through the point of contact of the inscribed sphere with the opposite face BCD. (2) The analogous property is true for a triangle ABC.

*Solution by Robert Bouvaist, Nantes, France.** Let (x_1, y_1, z_1, t_1) be the normal coördinates of an arbitrary point M with respect to $ABCD$ taken as the reference tetrahedron. The perpendiculars at A to the faces ACD, ADB, ABC meet the planes MCD, MDB, MBC, at A_1, A_2, A_3. Plane $A_1A_2A_3$ has as its equation

$$\begin{vmatrix} x & y & z & t \\ x_1 & y_1 & -y_1\cos a' & -y_1\cos b \\ x_1 & -z_1\cos a' & z_1 & -z_1\cos c \\ x_1 & -t_1\cos b & -t_1\cos c & t_1 \end{vmatrix} = 0,$$

where $a, \cdots$ designate the dihedral angles of edges $BC, \cdots$. In the above equation we may replace (x_1, y_1, z_1, t_1) by $(1, 1, 1, 1)$ since we are interested in the case where M is the incenter I. The line through A perpendicular to $A_1A_2A_3$ meets the plane BCD at $(0, 1+\cos c', 1+\cos b', 1+\cos a)$. These are precisely the coördinates of the foot of the perpendicular dropped from I on the plane BCD.

The analogous property for a triangle may be established by the same type of calculations or by elementary geometry.

Part (2) also solved by B. R. Leeds.

A Summation Problem

4305 [1948, 431]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$1+\left(\frac{1+\frac{1}{2}}{2}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}}{3}\right)^2+\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{4}\right)^2+\cdots=\frac{17\pi^4}{360}.$$

Solution by Martin Kneser, Tübingen, Germany. Noting that

$$\frac{1}{m}=\left(\frac{1}{m}-\frac{1}{m+n}\right)+\left(\frac{1}{m+n}-\frac{1}{m+2n}\right)+\left(\frac{1}{m+2n}-\frac{1}{m+3n}\right)+\cdots=\sum_{j=0}^{\infty}\frac{n}{(m+jn)(m+jn+n)},$$

* Translated by W. E. Byrne, Virginia Military Institute.

we get

$$\frac{1}{n}\sum_{m=1}^{n}\frac{1}{m}=\sum_{m=1}^{n}\sum_{j=0}^{\infty}\frac{1}{(m+jn)(m+jn+n)}=\sum_{k=1}^{\infty}\frac{1}{k(k+n)},$$

because $k=m+jn$ takes all values 1, 2, 3, $\cdots$ if j takes the values 0, 1, 2, $\cdots$ and m the values 1, 2, $\cdots$, n.

In the following all summations run from 1 to infinity unless indicated differently. If we denote the sum in question by S, then we have

$$\begin{aligned}
S&=\sum_{n}\left\{\frac{1}{n}\sum_{m=1}^{n}\frac{1}{m}\right\}^2=\sum_{n}\left\{\sum_{k}\frac{1}{k(k+n)}\right\}^2=\sum_{k,j,n}\frac{1}{jk(k+n)(j+n)}\\
&=\sum_{k<j}\frac{1}{jk(k+n)(j+n)}+\sum_{k=j}\frac{1}{jk(k+n)(j+n)}+\sum_{k>j}\frac{1}{jk(k+n)(j+n)}\\
&=\sum\frac{1}{k^2(k+n)^2}+2\sum_{k<j}\frac{1}{jk(k+n)(j+n)}\\
&=\sum_{k<j}\frac{1}{k^2j^2}+2\sum\frac{1}{k(k+m)(k+n)(k+m+n)}\\
&=\frac{1}{2}\left\{\sum\frac{1}{k^2j^2}-\sum_{k=j}\frac{1}{k^2j^2}\right\}+2\sum\left\{\frac{1}{kmn(k+m+n)}-\frac{1}{mn(k+m)(k+n)}\right\}\\
&=\frac{1}{2}\left\{\left(\sum\frac{1}{k^2}\right)^2-\sum\frac{1}{k^4}\right\}+2\sum\frac{1}{kmn(k+m+n)}-2S
\end{aligned}$$

according to the third expression for S. Since

$$\sum\frac{1}{k^2}=\frac{\pi^2}{6}\quad\text{and}\quad\sum\frac{1}{k^4}=\frac{\pi^4}{90}$$

we obtain

$$\begin{aligned}
\frac{1}{2}\left(3S-\frac{\pi^4}{120}\right)&=\sum\frac{1}{kmn(k+m+n)}=\sum\int_0^1\frac{x^{k+m+n-1}}{kmn}\,dx\\
&=\int_0^1\left(\sum\frac{x^k}{k}\right)^3\frac{dx}{x}=-\int_0^1\{\log(1-x)\}^3\frac{dx}{x}\\
&=-\int_0^1\{\log x\}^3\frac{dx}{1-x}=-\sum\int_0^1 x^{n-1}(\log x)^3dx\\
&=\sum\frac{6}{n^4}=\frac{\pi^4}{15},
\end{aligned}$$

whence $S=17\pi^4/360$, as required.

Also solved by the Proposer.

Power Residues Modulo p

4307 [1948, 432]. *Proposed by Emma Lehmer, Berkeley, California*

Let p be a prime number of the form $3\cdot 2^{\alpha}\cdot k+1$, and let n be any number not divisible by 3 or 2^{α}. Show that there are numbers a, b, independent of n, such that a, b, $a+1$, $b+1$ are nth power residues modulo p.

Solution by Martin Kneser, Tübingen, Germany. Euler's criterium states that the congruence $x^n\equiv c \bmod p$ is solvable if $c^{(p-1)/d}\equiv 1 \bmod p$, where $d=(n, p-1)$, and then it has d distinct solutions. We therefore must prove the existence of numbers a, b, such that a, b, $a+1$, $b+1$ are zeros of $x^{(p-1)/d}-1 \bmod p$ for all n.

It is sufficient to prove that they are zeros of $x^6-1 \bmod p$, because n is not divisible by 3 or 2, and hence 6 is a divisor of $(p-1)/d$. Now we have

$$\begin{aligned} x^6-1 &= (x^3+1)(x^3-1) = g(x)(x^2+x+1),\\ (x+1)^6-1 &= \{(x+1)^3-1\}\{(x+1)^3+1\}\\ &= h(x)\{(x+1)^2-(x+1)+1\} = h(x)(x^2+x+1). \end{aligned}$$

So, if a and b are zeros of $x^2+x+1 \bmod p$, then a, b, $a+1$, $b+1$ are zeros of $x^6-1 \bmod p$. Now $x^2+x+1=(x^3-1)/(x-1)$. According to Euler's criterium, $x^3\equiv 1 \bmod p$ has three distinct solutions and therefore $x^2+x+1\equiv 0 \bmod p$ has two distinct solutions, a and b.

Remark. Instead of $p=3\cdot 2^{\alpha}\cdot k+1$, we can assume $p=2^{\alpha}\cdot 3^{\beta}\cdot k+1$, if we suppose n not divisible by 2^{α} or 3^{β}.

Also solved by B. A. Hausmann and the Proposer.

4308 [1948, 432]. *Proposed by H. S. Wall, University of Texas*

Let $f(z)$ be a function of the complex variable z, such that

$$f(-w^2) = -\frac{F(w)}{w},$$

where $F(w)$ has the following properties: (a) $F(w)$ is an odd function of w, (b) $F(w)$ is analytic and $I[F(w)]\leqq 0$ for $I(w)>0$. Prove that $I[f(z)]\leqq 0$ for $I(z)>0$.

Solution by M. S. Robertson, Rutgers University. Put

$$w = i(1+t)/(1-t), \qquad F(w) = g(t).$$

Then $R[ig(t)]\geqq 0$ for $|t|<1$. The Poisson-Stieltjes representation in the Hergotz form* gives

$$ig(t) = \beta_0 i + \frac{\alpha_0}{2\pi}\int_0^{2\pi}\frac{1+te^{i\theta}}{1-te^{i\theta}}\,d\alpha(\theta), \qquad |t|<1, \tag{1}$$

* G. Herglotz, Leipziger Berichte, 1911, pp. 501–511.

where $ig(0)=\alpha_0+i\beta_0$, $\alpha_0 \geqq 0$, and $\alpha(\theta)$ is a non-decreasing bounded function in $(0, 2\pi)$. In terms of w, and using $F(-w)=-F(w)$, we obtain

$$F(w)=\frac{\alpha_0}{2\pi}\int_0^{2\pi}\frac{w\,d\alpha(\theta)}{w^2\sin^2\frac{\theta}{2}-\cos^2\frac{\theta}{2}}, \qquad I(w)>0 \tag{2}$$

with $\alpha_0=-I[F(i)]\geqq 0$. Setting $z=-w^2$, $|\arg z|<\pi$, we have

$$f(z)=f(-w^2)=\frac{-F(w)}{w}=\frac{\alpha_0}{2\pi}\int_0^{2\pi}\frac{d\alpha(\theta)}{1+(z-1)\sin^2\frac{\theta}{2}}. \tag{3}$$

Since $\alpha_0 \geqq 0$ it is seen from (3) that $I[F(z)] \leqq 0$ for $I(z)>0$.

It is of some interest to note that $f(1-z)$ is univalent for $|z|<1$ when $f(z)$ is not a constant. If we write

$$\frac{f(1-z)}{\alpha_0}=1+\sum_{n=1}^{\infty}c_n z^n, \qquad \alpha_0 \neq 0, \qquad |z|<1,$$

$$c_n=\frac{1}{2\pi}\int_0^{2\pi}\sin^{2n}\frac{\theta}{2}\,d\alpha(\theta),$$

we see that the sequence $\{c_n\}$ is totally monotonic, and thus by Fejér's test $f(1-z)$ is univalent for $|z|<1$.

The Tritangent Circles

4309 [1948, 432]. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Let ABC be a triangle, D, E, F the contact points of one of the tritangent circles with BC, CA, AB, respectively, and let A', B', C' and D', E', F' be the projections of a point M of that circle on BC, CA, AB and EF, FD, DE. Show that the lines joining the projections of M on $A'E'$ and $A'F'$, $B'F'$ and $B'D'$, $C'D'$ and $C'E'$ are concurrent.

Solution by the Proposer. The proposed property is a consequence of the following known property of the Simson line: If $JDEF$ is a quadrilateral inscribed in a circle, the projections of a point M of that circle on the Simson lines of M as to the four triangles formed by the vertices of the quadrilateral are on a straight line.

When J and D coincide, the Simson lines of M as to the triangles formed by D counted twice and E or by D counted twice and F are $A'F'$ and $A'E'$; hence the line joining the projections of M on $A'F'$ and $A'E'$ passes through the projection of M on the Simson line of M as to DEF.

This last point is the same when J coincides with E or F, and the theorem follows immediately.

Absolute Value of Polynomial and Derivative

4310 [1948, 504]. *Proposed by Paul Erdös, Syracuse University.*

Let $f(z)=z^n+\cdots$ be a polynomial of degree n. Denote by A_f the closed region (not necessarily connected) where $|f(z)|\leqq 1$. Prove that there always exists a z_0 in A_f with $|f'(z_0)|\geqq n$. Equality occurs only for z^n.

Solution by Robert Breusch, Amherst College, Massachusetts. Call C the boundary of A_f; thus $|f(z)|=1$ along C. Call $\overline{A}_f$ the (infinite) complement of A_f. $\overline{A}_f$ is connected and its boundary is C. Form

$$H(z)\equiv\frac{\{f'(z)\}^n}{\{f(z)\}^{n-1}}\equiv\frac{n^n z^{n(n-1)}+\cdots}{z^{n(n-1)}+\cdots}.$$

$H(z)$ is regular in $\overline{A}_f$; $H(\infty)=n^n$.

(1) Assume $|f'(z)|<n$ everywhere in A_f. Then $|H(z)|<n^n$ on C, and therefore $|H(z)|<n^n$ everywhere in $\overline{A}_f$, in contradiction to $H(\infty)=n^n$.

(2) Assume $|f'(z)|\leqq n$ everywhere in A_f. Then $|H(z)|\leqq n^n$ on C, and therefore (because of $H(\infty)=n^n$)$H(z)\equiv n^n$ in the whole plane. This means

$$\frac{1}{n}\{f(z)\}^{(1-n)/n}f'(z)\equiv 1,$$

whence

$$\{f(z)\}^{1/n}\equiv z+a,\qquad f(z)\equiv(z+a)^n.$$

Also solved by J. H. Curtiss, J. L. Ullman, and the Proposer who remarks that this is an easy corollary of no. 4229 [1948, 171].

Euler's Totient, Iteration

4311 [1948, 504]. *Proposed by V. L. Klee, Jr., University of Virginia*

If k and x are positive integers, let $f_k^1(x)=k\phi(x)$, where $\phi(x)$ is Euler's totient. For $j=2, 3, \cdots$, let

$$f_k^j(x)=f_k^{j-1}[f_k^1(x)].$$

Show that for $k\leqq 3$, the sequence, $f_k^1(x)$, $f_k^2(x)$, $\cdots$, is eventually constant, while for $k\geqq 4$, the sequence is eventually monotonically increasing.

Solution by Leo Moser, University of Manitoba. Let $f_k^j(x)=x_j$. If x_i contains any prime factor greater than k, then the operation $k\phi$ performed on x_i reduces the exponent of its largest prime factor by 1. Hence if k is fixed, there exists a j such that for all $i>j$, all the prime factors of x_i are less than or equal to k.

For $k=1$, for $i>j$, $x_i=1$.

For $k=2$, for $i>j$, $x_i=2^a$ and $2\phi(2^a)=2^a$.

For $k=3$, for $i>j$, $x_i=2^a3^b$ and $3\phi(2^a3^b)=2^a3^b$.

For $k\geqq 4$, for $i>j$,

$$k\phi(x_i) = kx_i \prod_{p \le k} (1 - 1/p) > kx_i \prod_{r=2}^{k} (1 - 1/r) = x_i,$$

whence $x_{i+1} > x_i$, which completes the proof.

Also solved by Paul Erdös, Free Jamison, and J. H. Wahab.

The Clairaut Differential Equation

4315 [1948, 586]. *Proposed by Albert Wilansky, Lehigh University*

Consider the Clairaut differential equation $y = px + f(p)$, where $p = y'$ and f has a derivative. Prove that if f' is monotone the singular solution has exactly one point in common with any particular solution.

Solution by S. H. Gould, Purdue University. The singular solution consists of the set of points (x, y) represented parametrically by

$$px - y + f(p) = 0, \qquad x + f'(p) = 0,$$

where $x = x(p)$, $y = y(p)$. A particular solution is represented by

$$cx - y + f(c) = 0,$$

with fixed c. These three equations have in common the point $x = -f'(c)$, $y = -cf'(c) + f(c)$, obtained by setting $p = c$. We now set $p = k \neq c$. If the equations are still consistent, their determinant vanishes:

$$\begin{vmatrix} k & -1 & f(k) \\ 1 & 0 & f'(k) \\ c & -1 & f(c) \end{vmatrix} = f(c) - f(k) + f'(k) \cdot (k - c) = 0.$$

Thus

$$\int_c^k f'(p)dp = f(k) - f(c) = f'(k) \cdot (k - c) = \int_c^k f'(k)dp,$$

which means, since $f'(p)$ is monotone, that $f'(p)$ is constant in $[c, k]$, so that $f'(k) = f'(c)$ and $f(k) = f(c) + f'(c) \cdot (k - c)$. But then

$$x(k) = -f'(k) = x(c) \quad \text{and} \quad y(k) = -kf'(k) + f(k) = y(c),$$

so that the common point $x(c)$, $y(c)$ is unique.

Also solved by E. S. Keeping, and Ingo Maddaus, Jr., and, assuming that also f' is strictly monotone, by Ta Li, Norman Miller, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y. and not to any of the other editors or officers of the Association.

Fundamental Science. By A. S. Eddington. Cambridge University Press, 1946. 8+292 pages. $6.00.

At the time of his death in November 1944, Sir Arthur Eddington left a manuscript which forms the basis of the present treatise "Fundamental Science." Although Eddington, had he lived, would have added an introduction to the book and also several more chapters one can consider the present work as complete in itself. Sir E. T. Whittaker, in the preface, points out that this book practically replaces all of Eddington's previous writings on his theory of the constants of Nature. The treatment of the theory in the first five chapters is essentially that used by Eddington in his Dublin lectures of 1943. The next three chapters, on sedenion analysis, contain work given in his book Relativity Theory of Protons and Electrons of 1936. In the last four chapters Eddington treats several new problems, including radiation, and the work of these chapters is essentially an addition to his previous published work on his new theory.

The aim and scope of the book is to provide a theory for the explanation of physical phenomena which range from the microscopic to the macroscopic. With such an aim in mind the reviewer feels that it would have been better to have written the book from the point of view of a deductive science. By this we mean that fundamental assumptions and definitions are clearly stated and listed and the main theory derived as consequence of these assumptions and definitions. Many times through the book Eddington seems to borrow results and concepts from quantum theory and one is never quite sure how these tie in with his new ideas. As a simple example we quote from the text:

"The term 'particle' survives in modern physics, but very little of its classical meaning remains. A particle can now best be defined as the conceptual carrier of a set of variates. We shall frequently use the term 'carrier' as an alternative to particle." In view of the fact that Eddington deems it necessary to define the term particle it is hard to see how it can be justified in using old concepts such as mass of a particle, energy of a particle, without a word of explanation of how these classical concepts are related to his new definition of a particle. This constant intermingling of new ideas and concepts and results obtained from modern quantum theory often obscures the justification for many of the statements and derivations given in the text.

How successful Eddington is in completing the aim and scope of his new theory the reviewer does not feel competent to judge. Certainly his theoretical calculations on the constants of Nature are in startling agreement with

observational results. In addition one feels that no work by a man of Eddington's stature in the scientific world could ever be lightly dismissed.

M. WYMAN

Advances in Applied Mechanics. Edited by R. von Mises and T. von Karman. (Vol. I). New York, Academic Press Inc. 8+293 pages. $6.80.

This book consists of six survey articles covering recent advances in applied mechanics and as such, is useful to both the expert and the beginner in any of the diversified fields included. Briefly, the subject matter is concerned with hydro-dynamics including boundary layer phenomena, turbulence, shock waves, compressible flows, elasticity, and problems in non-linear mechanics.

I. The first article by H. L. Dryden of the National Bureau of Standards is concerned with the mechanics of boundary layer flow and is largely non-mathematical and expository in character. While the author notes recent contributions to the mathematical concept of the boundary layer, including work on special flows, and the effects of compressibility on the laminar layer, the main part of the article reviews developments in the problem of stability of laminar flows—that is, the problem of the turbulence in the boundary layer. The recent experimental work (1940) of Schubauer and Skramstad of the National Bureau of Standards have confirmed and unified the theory of Heisenberg-Tolmien-Schlichting of small oscillations causing unstable waves and the theory of G. I. Taylor that free stream turbulence is the main factor in causing transition. s a result of the wealth of experimental data which is reproduced here and the completeness of the discussion this article is highly recommended as providing the reader with an over-all picture of the present state of boundary layer phenomena.

II. The second article by N. Minorsky of Stanford University is concerned with periodic solutions to non-linear differential equations of second order. It is a survey of the material contained in his book which has been published by Edwards Brothers, Ann Arbor, Michigan. Here, necessarily, the treatment is sketchy. But in spite of this, it is an extremely readable account of most of the important ideas in this field. These include the concept and classification of singular points to the differential equation, the concepts of limit cycles, indices and the well known theorems of the Benedixson concerning them, and Liapounoff stability. The analytical methods of solution which are discussed include those of Poincaré, Van der Pol and the Kryloff and Bogoliuboff adaption of the astronomical method of Lindstedt. The article concludes with a discussion of aspects of the problems of non-linear resonance including subharmonic resonance, entrainment of frequence, and parametric excitation. For those unacquainted with this field, this article should be most useful in enabling them to quickly grasp many of the essential ideas. On the other hand, with the publication of the Minorsky book mentioned above and the recently issued translation of "The Theory of Oscillations" by Andronow and Chaikin (Princeton

Univ. Press) most of the same material is available in English in a much more detailed fashion for the worker in this field.

III. The third article of this text is "Elasticity Papers Published in Holland, 1940–1946" by C. B. Biezeno. This article contains short reviews of a number of papers in theoretical and experimental elasticity. The theoretical papers are divided into the following groups: stability and buckling of shells and thin-walled cylinders; stresses in perforated plates; circular plates supported at a number of points regularly distributed along the boundary and rotatory-symmetrically loaded; miscellaneous papers dealing with the bending of beams, torsion of prismatic beams, the convergence of a special iterative process in structural analysis. Much attention is devoted to non-linear problems and their numerical solution which, for the most part, were designed to make the results available for all time. As such, someone contemplating work in this field would do well to scan this article before initiating any large scale calculations. A brief description of numerous experiments is furnished. Finally, two texts, "Technische Dynamik" by C. B. Biezeno and R. Grammel and "Plasticiteitsleer" by F. K. Th. van Iterson are discussed. Unlike the first two articles from which some ideas can be gleaned without too much previous experience in their fields, this article presupposes a thorough knowledge of the theory of elasticity.

IV. In the article "Theory of Turbulence" by J. M. Burgers, the author discusses a system of partial differential equations which he uses as a mathematical model for the theory of turbulence. In the author's terms, the difficulties encountered in the theory of turbulence are of a dual nature, the geometrical complexities introduced by the vectorial nature of the velocity and the complexities introduced by the non-linear terms. Independent variables are time t, and distance y; the dependent variables are U, the velocity of the primary motion, P the "analogue of the exterior force acting upon the primary motion," v the velocity of turbulence. First, the author shows that his system admits a first integral which is interpreted as consisting of various types of energies. Then, for the finite domain laminar solutions ($v=0$), and stationary turbulent solutions ($\partial v/\partial t=0$, $U=$constant) of the author's system are found. The latter involve the replacement of the original system by appropriate approximation equations. In particular, the author studies the spectrum of the solution for v and calculates the dissipation per component, *etc.* Finally, non-stationary solutions ($\partial v/\partial t\neq 0$, $U=$constant) are discussed for finite and infinite domains. Other developments include: a discussion of the correlation function; similarity conditions; systems with two components (u, w) in the secondary motion. The latter part of the article is devoted to a case in which the analogue of turbulent motion is allowed in two dimensions. The article is very readable and should prove of value to the novice in the field since it enables one to become acquainted with some of the important features of turbulent motion.

V. The article "Numerical Methods in Wave Interaction Problems" by H. Geiringer compares the author's and J. von Neumann's unpublished methods

of solving wave interaction problems. The calculations are carried out in detail for a specific simple boundary value problem in which Cauchy data are given at $t=0$ and side conditions are given along two straight walls. In the author's method, a single pressure-volume relation is assumed (weak shocks). The (x, t) plane is then divided into regions. Due to the simple boundary conditions, explicit formulas (due to Riemann) for the solution can be obtained in some of them. However, the principle difficulty lies in determining the solution in the zones beyond the unknown points where the shock line and rarefaction zone meet. The problem, here, is one of solving the partial differential equations of fluid dynamics when the boundary data satisfy certain relations (the shock conditions) along an unknown curve (the continuation of the straight shock line). To avoid this difficulty, the author makes use of a numerical method of essential successive approximations based on difference equations combined with interpolation. A slight modification of this scheme enables one to treat the case of strong shocks. Following this, von Neumann's mechanical model of shock motion and non-linear wave motion is described. Here it is imagined that the individual particles are connected by "springs" which are subject to non-linear laws of attraction and repulsion. Pictorial evidence is shown of the occurrence of oscillations in the solution where the shock line was previously determined by the author's method of calculation. All in all, this article is clearly written and provides an excellent résumé of many interesting results which came from the war effort in this direction.

VI. The sixth and final article by v. Mises and Schiffer provides a very readable account of Bergman's method of integration of Chaplygin's hodograph equation for the stream function of a steady irrotational flow. Chaplygin's basic idea was to make use of the properties of known incompressible flows in order (by analogy considerations) to deduce information about compressible flows. While this is possible in simple cases, there are many flow functions which are neither regular nor single-valued even for the incompressible case when they are expressed in the hodograph plane. Bergman's procedure is to first transform Chaplygin's equation into a form in which the stream function may be expanded in terms of an infinite series of products of harmonic functions and functions of a single variable. Recurrence relations are determined for the successive terms of this expansion, the series explicitly determined. It is shown to converge uniformly in a sector of the complex plane. However, the calculations are very long and tedious. In the second part of the paper, the authors (following Bergman) modify the basic partial differential equation, in order to obtain an easily computed expansion for the stream function. It is shown, in the remainder of the paper, that this implies use of a modified pressure-volume law. Graphical comparisons of the effects of this modified pressure-volume relation, the Karman-Tsien pressure-volume relation, and the pressure-volume relation for a real gas are furnished.

Because of the wide range of topics, the book requires the knowledge of diverse and specialized mathematical techniques on the part of the reader. More-

over the articles range from the first, which is purely expository, to the last, which is almost purely mathematical. Consequently, this book will serve different purposes for different readers.

C. L. DOLPH
N. COBURN

NEW BOOKS RECEIVED

The Theory of Probability, 2d Edition. By H. Reichenbach. Translated by E. H. Hutten and M. Reichenbach. Berkeley, University of California Press, 1949. 16+492 pp. $12.50.

Principles of a New Energy Mechanics. By Jakob Mandelker. Philosophical Library, New York, 1949. 73 pp. $3.75.

Tables of the Confluent Hypergeometric Function $F(n/2, 1/2; x)$ and Related Functions. (Applied Mathematics Series, No. 3.) Prepared by the Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards. Washington, U. S. Government Printing Office, 1949. 22+73 pp. $.35.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

CLUB REPORTS 1948–1949

Kappa Mu Epsilon, Central Missouri State College

The *Missouri Beta* chapter of *Kappa Mu Epsilon* held regular monthly meetings during the academic year 1948–49 and the summer session of 1949. The following talks were presented by members of the fraternity:

Geography and mathematics, by James Schmer
Pythagorean number triples, by Harold Woods
Calendars of the past and present, by Donna Lee Chitty
Calendrical computations, by Vol Russ
History and construction of magic squares, by Robert Boothe
Computational short-cuts and checks, by Quentin Smith and Keith Stumpff
Squaring the circle, by James Green
Trisecting the angle, by Myron Fitterling
Mathematical induction, by Edward Lowry
Boolean algebras, by Norton Jones
Inversion, by Marvin Goodman.

At the annual banquet in the Spring, Dr. E. J. Clark of the Education Division spoke on *The use of mathematics by the school counselor.* Nine new members were initiated into the fraternity. Eight members were previously initiated during a regular meeting in the Fall.

The members of our chapter journeyed to Central College, Fayette, Missouri in May to install the *Missouri Epsilon* chapter of *Kappa Mu Epsilon.* Dr. Claude H. Brown was the installing officer, and he was assisted by Miss Vonda Leah Lankford, Miss Donna Lee Chitty, Mr. Quentin Smith, and Mr. Robert Boothe, all of *Missouri Beta.*

Officers for 1949–50 are: President, James Green; Vice-President, Robert Boothe; Secretary, Donna Lee Chitty; Treasurer, Quentin Smith; Sponsor and Corresponding Secretary, Prof. L. W. Akers.

Pi Mu Epsilon, Oregon State College

At the five regular meetings of the *Oregon Beta* chapter of *Pi Mu Epsilon* the following programs were given during 1948–49:

The Laplace transformation, by Newton Smith
Paradoxes in infinite series, by O. Dale Smith
A graphometer, by Dr. Howard Eves
Happy mathematics, by James Price
Linkages, by J. Ericksen
Some solved and unsolved problems in geometry of sets of points, by A. S. Besicovitch of Cambridge University, England.

Early in the school year, *Pi Mu Epsilon* sponsored a joint picnic with the mathematics faculty. Another social event of the year's activities was the initiation of fifty-one new members in May. At the banquet held after the initiation, Dr. R. A. Rosenbaum of Reed College, Portland, Oregon, gave a talk on *Some characteristic properties of a circle.*

The officers for the academic year 1949–50 are: Director, R. D. Oeder; Vice-Director, O. Dale Smith; Secretary, R. I. Daniels; Treasurer, Prof. G. A. Williams.

Delta X, University of Toledo

The following papers were presented during the twentieth annual program of *Delta X*, Mathematics Club of the University of Toledo:

Air force research, by Dr. Carroll Amos
Mathematical puzzles and paradoxes, by Dr. Howard Swift
The trisection of the angle, by Richard Castanias
A problem in Diophantine equations, by Douglas Lang
Graphical analysis, by Roger Woodruff
History and computation of pi, by John Suprock
Relativity, by Jack Scarlett
Nomograms, by Richard Reisbach.

Each of the regular meetings concluded with a social period at which refreshments were served. *Delta X* was host to the Maumee High School *Mathematics Club* at one meeting and entertained the *Icosahedron Club* at a Christmas party. The social activities also included an annual banquet and a picnic.

The officers for 1949–50 are: President, Jack Scarlett; Vice-President, George Halgas; Secretary-Treasurer, Anna Tom.

Mathematics Club, Regis College

The annual record of activities of the Regis College *Mathematics Club* for 1948–49 include the following papers:

Shadows, by Anna McFarlane
Approximate construction for e, by Phyllis Moran
Lewis Carroll, mathematician, by Katherine Healey
Moebius strip, by Ann McDonnell
Mathematics in music, by Florence Seaver
Mathematical recreations, by Claire Sweeney
Astronomy, by Ann McDonnell
The abacus, by Ann Downey
Poets' numbers, by Katherine Healey
Peaucellier cells, by Margaret Linney
Architectural wonders of the world, by Barbara Lane
Was man made by Earth or was Earth made for man?, by Louise Kelley
Mathematical designs, by Mary White
Opportunities for mathematically trained college graduates
Use of mathematics in aircraft industry, by Maryann Boyce.

The program included a Christmas party, a supper party at which members of the class of '49 were guests, and a tea at which the Mathematics Alumnae announced the winner of a $25.00 award for the graduate from the department who had earned the highest average in mathematics over a period of four years.

The entire club attended the meeting of the *Greater Boston Mathematics Club* which was held at Harvard College. There was also an attendance of delegates at meetings of the Association of Teachers of Mathematics of New England.

Our projects included *From Near and Far*, a pin map displaying the geographic location from which Regis College students come and to which Regis Alumnae go. A second graph in the College Display Case depicts the survey of the positions which our Alumnae hold and discloses a summary of the vocations in which the Regis College Alumnae are working. Ingenious constructions in balsa have been added to the model collection including two views of the tesseract and simplex.

Officers for 1948–49 were: President, Barbara Lane; Vice-President, Katherine Healy; Secretary, Maryann Boyce; Treasurer, Virginia Lee.

Mathematics Club, Kansas State College

During the academic year 1948–49, the following papers were presented at meetings of the *Mathematics Club* of Kansas State College:

A new look at an old formula, by Prof. F. Sloat
A history of the exponent, by V. Bly
Graphic solution of equations, by K. Loewen
Matrices, by Miss Doris Lloyd
Mathematics and the imagination, by A. Lloyd
Celestial navigation, by L. Trapp
A history of the calculus of variations, by Prof. R. Sanger
Symmetries of the square. by W. Cowell.

Mr. Bly's paper was subsequently published in the February 1949 issue of the *Bulletin of the Kansas Association of Teachers of Mathematics.*

Officers for 1948–49 were: President, W. Cowell; Vice-President, V. Bly; Secretary-Treasurer, A. Lloyd; Faculty Advisor, Prof. P. Young.

Pi Mu Epsilon, Marquette University

The following papers were presented to the members of the *Wisconsin Alpha* chapter of *Pi Mu Epsilon* during 1948–49:

How to draw a straight line, by Dr. H. Pettit
Symmetrical coordinates, by Dr. S. Canterbury
A physicist looks at mathematics, by Dr. G. Barkow
Astronomy, by Mr. Edward Halbach
Mathematical education in central Europe, by Dr. J. Talacko
Sir Isaac Newton, by Dr. G. Parkinson.

A Christmas party and an annual banquet were held during the year.

Officers elected for 1949–50 are: Director, William Golomski; Vice-Director, William Weideman; Recording Secretary, Kathleen Murphy; Corresponding Secretary, Virginia Higgins; Treasurer, Robert Johnson; Librarian, Harry Hesse.

Kappa Mu Epsilon, Southwest Missouri State College

The *Missouri Alpha* chapter of *Kappa Mu Epsilon* at Southwest Missouri State College has an enrollment of 26 members and participated in five meetings during the academic year of 1948–49. The following papers were presented:

A finite geometry of twenty-five points, by Edward Rykowski and Carl Gabriel
A photographic method of portraying cycloidal loci, by L. Shiflett
Schwartz's problem, by Julius Komarmy
The sundial, by Dr. V. Whitney

The first paper listed above also was presented by Mr. Rykowski at the

national biennial convention of *Kappa Mu Epsilon* at Washburn University, Topeka, Kansas, in April.

Officers for 1949–50 are: President, Don Barbarick; Vice-President, Kenneth Mustain; Secretary, Mary Jane Robinette; Treasurer, James Jakobsen; Corresponding Secretary, Harold Skelton; Sponsor, Prof. P. Pummill.

Kappa Mu Epsilon, Coker College

The *South Carolina Alpha* chapter of *Kappa Mu Epsilon* held regular monthly meetings during the school year 1948–49. The programs were composed entirely of discussions conducted by various students on current mathematical problems and puzzles. A number of problems proposed in the PENTAGON and the MONTHLY were reviewed, and some of the members turned in solutions to these problems. For next year a more varied program of papers is being planned.

The officers for 1949–50 are: President, Shirley Jenkins; Secretary, Betty Rose Reeves; Faculty Advisor, Frank W. Saunders.

Mathematics Club, Upsala College

The *Mathematics Club* of Upsala College held a Christmas party and a mid-year festival jointly with *Kappa Mu Epsilon*, and a buffet supper at the home of one of the members.

The Club had bimonthly meetings. The plan was to study a famous personality by investigating his mathematical environment and some of his contributions. The following papers were presented:

The life of Blaise Pascal, by James Gill

The Pascal triangle, by Martin Monroe

The life of Gauss, by John Armstrong

Gauss' law of quadratic reciprocity, by Robert Reed

Development of Gauss' theorem pertaining to the electric intensity about any closed surface, by Robert Warnken

The life and contributions of Zeno, by Seymour Hersh, William Manco, and Edward Pulchlopek

The life of Napier, by Lloyd Thornton

Napier's rods, by Frances Rischmuller

Evariste Galois, by Hannah Magill

The theory of fields and groups, by Jerome Drexler

The life and works of Pierre de Fermat, by Dorothy Fischbeck and William Stachel

Infinite series, by Ethel Larson

Mathematical truths, by James Christakos

Mathematical fallacies, by Jack Presser

The invention of analytical geometry, by Clair Willhelm

The squaring of a circle, by Robert Schenk and Bernt Benson.

The newly elected officers for the year 1949–50 are: President, James Christakos; Vice-President, William Stachel; Secretary, Dorothy Fischbeck; Treasurer, Richard Lawrence.

Mathematics Club, University of Wisconsin

Following is a list of the talks given at the various *Mathematics Club* meetings at the University of Wisconsin:

On the approximation of a function by a power series, by Prof. Max Dehn

Convex functionals, by Prof. W. E. Eberlein

Affine invariants of a pair of hypersurfaces, by Dr. C. C. Hsiung

Elastic stability of plywood plates and shells, by Prof. H. W. March

Multiple Fourier series, by Prof. Lamberto Cesari of the University of Bologna

On certain classes of algebraic integers, by Dr. J. B. Kelly

The topology of function spaces, by Prof. Casimir Kuratowski of the University of Warsaw and the Institute for Advanced Study

Generalized curves, by Prof. L. C. Young

Asymptotic solutions of a second order differential equation, by Dr. E. D. Cashwell

Undecidable sentences in set theory and real numbers arithmetic, by Prof. Andrzej Mostowski of the University of Warsaw and the Institute for Advanced Study

Development of the axiomatic definition of set and Skolem's paradox, by Dr. Alfons M. F. Borgers of the Belgian National Foundation for Scientific Research

Paracompact spaces, by Dr. C. H. Dowker of Princeton University

The generalization of the Moebius mu-function to a partially ordered set, by Mr. A. F. Strehler

Integral valued entire functions, by Dr. R. C. Buck of Brown University.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1950:

Columbia University, Teachers College. July 10 to August 8: Professor Clark, teaching arithmetic in the elementary school; Professor Fehr, current problems in teaching secondary school mathematics, professionalized subject matter in advanced secondary school mathematics; Professor Shuster, modern business arithmetic, teaching algebra in secondary schools; Professor Snader, teaching

geometry in secondary schools, history of mathematics; Professor Yates, field work in mathematics, applications of mathematics.

Duke University. June 14 to July 1: Professor Dressel, solid analytic geometry. July 6 to August 12: Professor Carlitz, finite differences, thesis seminar; Professor Gergen, mathematical statistics, theory of functions. August 14 to August 31: Professor Roberts, theory of functions.

Ohio State University. June 20 to September 1: Professor Reichelderfer, advanced calculus, dimension theory; Professor Miller, solid analytical geometry, fundamental ideas in algebra and geometry; Professor Rechard, advanced calculus, differential equations; Professor Cashwell, integral equations.

State University of Iowa. June 12 to August 9: Professor Knowler, elements of statistics, actuarial aspects of casualty insurance; Professor Wylie, astronomy; Professor Smiley, synthetic projective geometry, introduction to advanced mathematics; Professor Woods, study in secondary geometry, pure geometry; Professor Conkwright, differential equations; Professor Oberg, elementary theoretical mechanics, Fourier and allied series; Professor Price, the supervision of mathematics; Professor Muhly, matrices and determinants, elementary theory of numbers; Professor Cosby, advanced calculus, calculus of variations.

Syracuse University. July 5 to August 11: infinite series; intermediate course in algebra; introduction to modern mathematics; history of mathematics; aids in the teaching of high school mathematics.

University of Buffalo. July 5 to August 12: Professor Gehman, infinite series; Professor Montague, algebras and their arithmetics, geometrical constructions; Professor Schneckenburger, elementary finite groups.

University of California at Los Angeles. June 19 to August 11: Professor Bell, workshop in secondary school mathematics; Professor Daus, geometrical constructions; Professor Hedlund, differential geometry in the large; Professor Horn, advanced calculus; Professor Sokolnikoff, partial differential equations; Professor Taylor, functions of a complex variable; Dr. Wing, vector analysis; Professor Zelinsky, theory of fields.

University of Colorado. June 19 to July 21 and July 24 to August 25: Professor Kurt Mahler, University of Manchester, England, geometry of numbers; Mr. Charlesworth, advanced course in teaching of mathematics (first term); Professor Farnell, higher algebra (second term), calculus of variations; Mr. Hunt, elementary differential equations; Professor Hutchinson, Fourier series and boundary value problems; Professor Jones, higher algebra (first term), elementary topology (first term); Professor Kendall, selected topics in geometry (second term).

University of Detroit. June 26 to August 4: Professor Smith, theory of equations, differential geometry; Professor McCarthy, advanced calculus, Fourier series and harmonic analysis; Professor Mehlenbacher, intermediate differential equations.

University of Illinois. June 20 to August 19: Professor Bourgin, theory of rings; Professor Day, functions of real variables; Professor Ketchum, theory of

functions of a complex variable; Professor Landin, projective geometry; Professor Munroe, measure and integration; Staff, fundamental concepts of mathematics; introduction to higher algebra; introduction to higher geometry; introduction to higher analysis, complex variables; mathematical methods in engineering and science; introduction to higher analysis, real variables.

University of Michigan. June 26 to August 19. Professor Eckmann, general spaces, combinatorial methods; Professor Hildebrandt, theory of differential equations; Professor Myers, differential geometry; Professor Carver, theory of statistics II, finite differences; Professor Craig, multivariate analysis; Professor Dwyer, analysis of variance, statistical and computational methods with punched cards; Professor Dushnik, foundations of mathematics; Professor Fischer, intermediate mathematics of life insurance; Professor Hay, advanced mechanics, vector analysis; Professor Rothe, application of functions of a complex variable, topics in mathematical physics; Professor Thrall, higher algebra, theory of matrices; Professor Bartels, Fourier series; Professor Coburn, dynamics; Professor Dolph, operational mathematics; Professor Jones, teaching of algebra, history of geometry; Professor Reade, functions of a complex variable, harmonic and subharmonic functions; Professor Young, advanced calculus; Dr. Darling, theory of probability, theory of statistics I; Dr. Leisenring, synthetic projective geometry.

University of Nebraska. June 6 to July 29: Professor Basoco, differential equations, solid analytic geometry; Professor Leavitt, advanced calculus; Professor Halfar, fundamental concepts of mathematics; Professor Camp, analysis.

University of North Carolina. June 12 to July 20: Professor Cameron, elementary algebra from an advanced viewpoint; Professor Garner, history of mathematics; Professor Hill, elementary mathematical statistics; Dr. Mann, advanced calculus; Professor Brauer, advanced number theory of rational integers; Professor Brown, algebraic invariants. July 21 to August 29: Professor Mackie, theory of equations; Professor Hoyle, differential equations; Professor Whyburn, advanced calculus II; Professor Hershner, advanced theory of functions of a complex variable; Professor Lasley, differential geometry.

University of Oregon. June 19 to August 11: Professors Civin and Niven, algebra for teachers; Professor Ghent, geometry for teachers, foundations of mathematics for teachers; Professor Moursund, selected topics (pure mathematics); Professor Civin, transformations of Fourier and Laplace.

University of Wisconsin. June 26 to August 18: Professor Bing, advanced analytic geometry, higher mathematics for engineers; Professor Mayor, topics in geometry; Professor Sokolnikoff, higher mathematics for engineers; Professor Owens, higher analysis; Professor Young, advanced calculus, divergent series; Professor Mark, advanced calculus; Professor Arnold, mathematics of elementary statistics; Professor MacDuffee, survey of the foundations of arithmetic, tensor analysis; Professor Colvin, mathematical applications; Professor Langer, harmonic analysis; Staff, determinants and matrices, higher algebra.

University of Wyoming. June 19 to July 21: Professor Barr, numerical analysis; Professor Neubauer, history of mathematics; Professor Schwid, advanced calculus, theory of functions of a complex variable; Professor Smith, ordinary differential equations; Professor Varineau, theory of equations, abstract algebra. July 24 to August 25: Professor Barr, vector analysis; Professor Schwid, seminar in analysis, college geometry, theory of functions of a complex variable; Professor Smith, partial differential equations; Professor Varineau, higher algebra, fundamental concepts of mathematics, abstract algebra.

PERSONAL ITEMS

Dr. Ilse L. Novak of Wellesley College has been awarded one of the Frank B. Jewett postdoctoral fellowships for 1950–51 by the Bell Telephone Laboratories.

The Duodecimal Society of America announces that the following officers have been elected for 1950: Chairman of the Board, F. E. Andrews; President, H. C. Robert, Jr.; Vice-Presidents, P. E. Friedemann and Dr. Nathan Lazar; Editor, G. S. Terry; Secretary, R. H. Beard; Treasurer, H. K. Humphrey.

Harvard University reports: Professor Georges de Rham of the University of Lausanne and the University of Geneva was Visiting Professor for the fall term, 1949; Professor W. V. D. Hodge of Cambridge University is Visiting Professor during the spring term, 1950.

Iowa State College announces the appointments of the following instructors: Mr. A. M. Feyerherm, Mr. R. N. Goss and Mr. R. A. Worsing.

At Miami University: Dr. W. C. Sangren, formerly an Atomic Research Fellow at the University of Michigan, has been appointed to an assistant professorship; Instructor Alberta L. Wolfe has been promoted to an assistant professorship; Graduate Assistants Kathryn B. Aldrich and R. H. Johnston have been promoted to instructorships.

Oklahoma Agricultural and Mechanical College announces: Dr. C. J. Clark of Ohio State University has been appointed Instructor and Research Assistant; Dr. Arthur Zeichner, formerly research associate at Harvard University, has been appointed Instructor and Research Assistant; Associate Professor Joseph Barnett, Jr., has retired with the title of Associate Professor Emeritus; Associate Professor C. E. Marshall and Assistant Professor J. V. Robison have returned to the staff.

Pennsylvania State College makes the following announcements: Dr. E. J. Akutowicz, formerly at the Institute for Advanced Study, has been appointed to an assistant professorship; Dr. Nathan Jaspen has been appointed Lecturer; Mr. A. J. Goldstein, Mr. J. D. E. Konhauser and Mr. E. F. Ormsby have been appointed to part-time instructorships; Miss Alice M. Hanna has been appointed graduate assistant; Professor Orrin Frink has been promoted to the position of Head of the Department of Mathematics; Professor F. W. Owens is now Professor Emeritus; Assistant Professor Helen B. Owens has retired.

At Rensselaer Polytechnic Institute: Dr. M. R. Spiegel, previously graduate

assistant at Cornell University, and Mr. J. C. Gibson, graduate assistant at Yale University, have been appointed to instructorships; Assistant Professor R. F. Tessier has retired.

Texas Technological College announces the following: Professor J. M. Michie, formerly head of the Department of Mathematics and Astronomy, and Professor E. L. Thompson retired on February 1, 1950.

Tulane University makes the following announcements: Dr. Sze Tsen Hu of Academia Sinica, Shanghai, China, has been appointed Lecturer; Dr. Herman Cohen of the University of Wisconsin has been appointed to an instructorship; Mr. J. W. Ellis has been appointed Atomic Energy Commission Fellow.

University of Illinois reports: Dr. Herman Chernoff, formerly statistician for the Cowles Commission, University of Chicago, and Instructor Harry Lass of the University of California have been appointed to assistant professorships; Dr. Meyer Jersion has been appointed Research Instructor; Associate Professor M. M. Day has been promoted to a professorship; Instructors W. F. Atchison, M. K. Fort, Jr., F. E. Hohn, and E. J. Scott have been promoted to assistant professorships; Assistant Clarence Phillips has been promoted to an instructorship.

At the University of Kentucky: Miss H. Fay Hays and Mr. Sherman Vanaman have been appointed to graduate assistantships; Mrs. Lydia R. Fischer has retired.

The University of Southern California reports the following appointments: Dr. L. A. Henkin of Princeton University to an assistant professorship; Dr. Harold Shniad, Purdue University, to an instructorship.

University of Washington makes the following announcements: Mr. G. E. Uhrich has been appointed Acting Instructor; Assistant Professor Mary E. Haller has been promoted to an associate professorship; Dr. Fumio Yagi has been promoted to an assistant professorship.

University of Wyoming reports the following appointments: Associate Professor R. L. Calvert of Utah State Agricultural College to an assistant professorship; Mr. J. A. Weiss, formerly research chemist, Glenn L. Martin Company, Baltimore, Ohio, to an instructorship.

Utah State Agricultural College announces the following appointments to instructorships: Mr. J. K. Everton, Mr. J. E. Faulkner, Mr. Karl Koerner, Mr. P. A. Madden, Mr. D. W. Stoddard and Mr. J. M. Ward.

At Virginia State College: Associate Professor H. M. Linette of Virginia Union University has been appointed to an associate professorship; Miss Dorothy M. Cheek has been appointed Instructor in Mathematical Education; Mrs. Bertrice W. King has been promoted to an assistant professorship.

Washburn University announces the following appointments to instructorships: Mr. M. C. Hoover and Mr. H. Q. Rolfs of the University of Kansas.

Wayne University reports: Mr. Milton Sobel has been appointed to an assistant professorship; Assistant Professor R. L. Ackoff has been granted a leave of absence for six months so that he may serve as consultant for the United States Bureau of Census.

At West Virginia University: Miss Elizabeth H. McLaughlin and Mr. Peter Zaphyr have been appointed to instructorships; Assistant Professor M. L. Vest has been promoted to an associate professorship; Assistant Professor I. D. Peters is on leave for the year and is studying at Columbia University.

Yale University makes the following announcements: Dr. Shizuo Kakutani of the Institute for Advanced Study has been appointed to an assistant professorship; Dr. R. R. Benard of the University of Virginia and Dr. D. T. Finkbeiner, California Institute of Technology, have been appointed to instructorships; Associate Professor Nathan Jacobson has been promoted to a professorship; Dr. D. F. Votaw has been promoted to an assistant professorship; Professor W. R. Longley has retired with the title of James E. English Professor Emeritus of Mathematics.

Dr. M. M. Andrew who has been engaged in operational research with the Naval Operations Evaluation Group has joined the staff of the machine development section of the applied mathematics laboratories, National Bureau of Standards.

Miss Betty J. Barnett, graduate assistant at the University of Kentucky, has received an appointment at Sue Bennett College.

Miss Adrian M. Baucom, graduate assistant at the University of Kentucky, is teaching at Kenton High School, Kenton, Tennessee.

Miss Dorothy De Witt has been appointed to an instructorship at Iowa State Teachers College.

Professor Patric DuVal of the University of Istanbul, Turkey, has been appointed Professor of Mathematics at the University of Georgia.

Assistant Professor W. C. G. Fraser of Rensselaer Polytechnic Institute has accepted an assistant professorship at Dartmouth College.

Mr. R. P. Hobbs is now Vice-President of Rinehart & Company, Incorporated.

Dr. Nathan Lazar, part-time instructor at Teachers College, Columbia University, has been appointed Associate Professor of Education at Ohio State University.

Mr. L. B. McBride, formerly instructor at Utah State Agricultural College, is teaching at Shelley High School, Shelley, Idaho.

Mr. Jacques Saint-Pierre of the University of Montreal has been promoted to the position of assistant lecturer.

Mr. F. W. Stallard has been appointed to an instructorship at East Tennessee State College.

Assistant Professor E. C. Stopher, Jr., of Miami University has been appointed to a professorship at Fort Hays Kansas State College.

Mr. J. A. Wilson of Case Institute of Technology has received an appointment as assistant professor at Baldwin-Wallace College.

Professor L. A. H. Warren of the University of Manitoba died on October 6, 1949. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred and six persons have been elected to membership on applications duly certified.

M. W. AL-DHAHIR, B.A. (Baghdad) Student Columbia University, New York, N. Y.

SHOLOM ARZT, M.S. (N.Y.U.) Instructor, New York University, N. Y.

R. P. BACON, A.B. (Miami U.) Grad. Asst., Miami University, Oxford, Ohio.

BARBARA BAKER, A.B. (Brown) Research Asst., Lehigh University, Bethlehem, Pa.

NATHAN BAROTZ, B.S.I.E. (N.Y.U.) Instructor, Brooklyn College, N. Y.

H. S. BERG, B.S. (North Dakota) Grad. Asst., University of North Dakota, Grand Forks, N. D.

L. G. CAMPBELL, A.M. (New Jersey S.T.C.) Instructor, New Jersey State Teachers College, Upper Montclair, N. J.

J. H. CASE, Student, Alabama Polytechnic Institute, Auburn, Ala.

HAROLD CHATLAND, Ph.D. (Chicago) Professor, Montana University, Missoula, Mont.

ERMINE A. CHRISTIAN, A.B. (Mercer) Mathematician, Naval Ordnance Laboratory, Silver Spring, Md.

A. C. CICCHESE, B.S. (Notre Dame) Grad. Student, St. Mary's University, San Antonio, Tex.

D. W. COOK, B.S. (California) Senior Instructor, California State Polytechnic College, San Luis Obispo, Calif.

MAX CORAL, Ph.D. (Chicago) Asso. Professor, Wayne University, Detroit, Mich.

W. D. COWAN, M.A.E. (Florida) Instructor, Chipola Junior College, Marianna, Fla.

M. K. COX, B.S.I.E. (V.P.I.) Assistant, Kent State University, Ohio.

S. H. CUNHA, B.Sc. (McGill) Electrial Engineer, Quebec Hydro-Electric Commission, Montreal.

P. M. CURRAN, M.A. (Columbia) Instructor, Fordham University, New York, N. Y.

GEORGIE T. DAVIS, M.S. (V.P.I.) Instructor, Virginia Polytechnic Institute, Blacksburg, Va.

D. E. DEAL, A.M. (Michigan) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

H. W. DOSS, JR., M.A. (Missouri) Instructor, University of Omaha, Nebr.

F. J. DUARTE, Ing.Civ. (Caracas) Director de Fronteras, Ministerio de Relaciones Exteriores, Caracas, Venezuela.

G. R. DUBE, B.S. (M.I.T.) Assistant, Yale University, New Haven, Conn.

D. L. DUNNING, B.A. (North Central) Grad. Asst., Kent State University, Ohio.

SIDNEY EDELSON, M.A. (N.Y.U.) Mathematician, U. S. Naval Observatory, Washington, D. C.

BERNARD EISENBERG, B.E.E. (Brooklyn Poly.) Teaching Assistant, Rutgers University, New Brunswick, N. J.

JEROME EMERSON, Student, University of Buffalo, N. Y.

A. E. FOSTER, M.A. (Tennessee) Instructor, University of Kentucky, Lexington, Ky.

ABRAHAM FRANCK, Ph.D. (Minnesota) Lecturer, University of Minnesota, Minneapolis, Minn.

D. E. FREELAND, Student, Northwestern University, Evanston, Ill.

E. L. FRIEDMAN, Student, Massachusetts Institute of Technology, Cambridge, Mass.

J. L. GAMMEL, Ph.D. (Cornell) Teaching Assistant, Cornell University, Ithaca, N. Y.

S. I. GASS, M. A. (Boston U.) Mathematician Aberdeen Bombing Mission, Computing Unit, Los Angeles, Calif.

R. W. GEIS, M.A. (Northwestern) Captain, U. S. Army, Fort Bliss, Tex.

J. C. GIBSON, M.A. (Yale) Instructor, Rensselaer Polytechnic Institute, Troy, N. Y.

A. G. GRACE, JR., B.S. (Trinity) Instructor, Trinity College, Hartford, Conn.

J. B. GREGG, M.A. (National Northern) 2022 Tremont Road, Columbus, Ohio.

R. B. GREKILA, B.S. (Ohio U.) Grad. Asst., University of Maryland, College Park, Md.

F. D. GROGAN, B.S. (Louisiana Poly.) Teaching Assistant, University of Oklahoma, Norman, Okla.

EDWIN HALFAR, Ph.D. (Iowa) Asst. Professor, University of Nebraska, Lincoln, Nebr.

J. S. HOKANSON, A.M. (Michigan) Acting Instructor, Ohio University, Athens, Ohio.

J. G. HOWCROFT, A.B. (Ohio) Teaching Fellow, West Virginia University, Morgantown, W. Va.

G. H. JAEGER, M.A. (Minnesota) Instructor, Macalester College, St. Paul, Minn.

R. E. JANSSEN, JR., M.A. (Illinois) Instructor, University of Utah, Salt Lake City, Utah.

G. E. KALDENBERG, M.S. (Iowa S.C.) Asst. Professor, Iowa State College, Ames, Iowa.

KARLIS KAUFMANIS, Magister Math. Sci. (Latvia) Asst. Professor, Gustavus Adolphus College, St. Peter, Minn.

M. R. KENNER, B.A. (Rochester) Grad. Student, University of Minnesota, Minneapolis, Minn.

NELLIE M. KITCHENS, M.A. (Missouri) Head of Mathematics Department, Hickman High School, Columbia, Mo.

W. B. KOENEN, Student, Macalester College, St. Paul, Minn.

D. H. KRAFT, M.A. (Vanderbilt) Student, Ohio State University, Columbus, Ohio.

SEYMOUR LANDAU, B.A. (N.Y.U.) Instructor, University of Connecticut, Storrs, Conn.

B. F. LAPOSKY, Draftsman, Laposky Drafting Service, Cherokee, Iowa.

VIRGINIA M. LARSON, M.A. (Kent) Instructor Stratford College, Danville, Va.

J. R. LEE, M.A. (Yale) Student, Yale University, New Haven, Conn.

D. R. LINTVEDT, M.S. (Wisconsin) Asst. Professor, Upsala College, East Orange, N. J.

A. J. LOHWATER, Sc.M. (Rochester) Instructor, University of Michigan, Ann Arbor, Mich.

JAYWOOD LUKENS, LL.B. (Fordham) Asso. Actuary, New York, N. Y.

HENRY MANN, B.A. (Brooklyn Coll.) Student, Brooklyn College, N. Y.

S. T. MARTIN, B.A. (Hampden-Sidney) Acting Head of Mathematics Department, Newberry College, S. C.

A. E. MAY, Ph.D. (Wisconsin) Assistant Director, University of Wisconsin, Racine Extension Center, Wis.

G. T. MCCREADY, B.S. (U. S. Naval Acad.) Captain, U. S. Navy, Long Beach, Calif.

J. E. MORGAN, Student, St. Mary's University, San Antonio, Tex.

K. E. MORGAN, Ed.D. (Teachers Coll., Columbia) Instructor, Mt. Kisco High School. N. Y.

SABRINA MORLAN, M.S. (Michigan) Instructor, Kansas State College, Manhattan, Kans.

D. R. MORRISON, Ph.M. (Wisconsin) Grad. Asst., University of Wisconsin, Madison, Wis.

R. D. OEDER, B.A. (Willamette) Grad. Asst., Oregon State College, Corvallis, Ore.

N. G. PARKE, III, Ph.D. (M.I.T.) Consulting Physicist, Concord, Mass.

K. S. PHELAN, M.A. (Columbia) Instructor, St. John's University, Brooklyn, N. Y.

T. J. PIGNANI, M.S. (Bucknell) Student, University of North Carolina, Chapel Hill, N. C.

E. J. POLAK, Asst. Instructor, Bucknell University, Lewisburg, Pa.

P. L. POSTON, M.B.A. (Michigan) Actuary, Great Lakes Mutual Life Insurance Co., Detroit, Mich.

J. F. PRICE, Ph.D. (Oregon S.C.) Asst. Professor, Oregon State College, Corvallis, Ore.

E. A. RANDICH, M.S. (Pittsburgh) Development Engineer, United Engineering & Foundry Co., Pittsburgh, Pa.

R. W. RAYMOND, M.S. (Iowa) Instructor, St. Ambrose College, Davenport, Iowa.

R. B. REISEL, B.S. (DePaul) Student, University of Chicago, Ill.

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SARA L. RIPY, M.A. (Kentucky) Instructor, University of Kentucky, Lexington, Ky.

H. E. ROBBINS, Ph.D. (Harvard) Asso. Professor, University of North Carolina, Chapel Hill, N. C.

MARCIA C. SAILE, A.M. (Michigan) Instructor, West Virginia University, Morgantown, W. Va.

JEAN E. SAMMET, M.A. (Illinois) Assistant, University of Illinois, Urbana, Ill.

H. A. SAYLES, 106 North Grove Street, East Orange, N. J.

J. A. SCHATZ, B.S.E.E.(V.P.I.) Student, Brown University, Providence, R. I.

SEYMOUR SCHUSTER, M.A.(Columbia) Instructor, Pennsylvania State College, Pa.

R. L. SCHWALLER, B.S.(St. Norbert) Student, St. Norbert College, West De Pere, Wis.

R. J. SEMPLE, Student, University of Toronto, Ont.

K. J. SIDEBOTTOM, B.S.(Indiana Central Coll.) Instructor, Indiana Central College, Indianapolis, Ind.

SISTER M. LEONA, M.S.(Notre Dame) Instructor, Mercy College, Detroit, Mich.

ABE SKLAR, M.S.(Chicago) Grad. Student, University of Chicago, Ill.

C. G. SOLKY, Student, New York University, N. Y.

M. E. SPERLINE, B.S.(Kansas S.C.) Instructor, University of Colorado, Boulder, Colo.

F. W. STALLARD, M.S.(Brown) Instructor, East Tennessee State College, Johnson City, Tenn.

W. G. STOKES, M.A.(Sam Houston S.T.C.) Instructor, Navarro Junior College, Corsicana, Tex.

E. C. STOPHER, Ph.D.(Iowa) Professor, Fort Hays Kansas State College, Kans.

O. E. STRONG, M.S.(Indiana S.T.C.) Asso. Professor, Indiana State Teachers College, Terre Haute, Ind.

O. R. TAYLOR, M.S.(Oklahoma A. & M.) Instructor, St. Ambrose College, Davenport, Iowa.

J. G. TORIAN, Student, St. Mary's University, San Antonio, Tex.

C. A. TRUESDELL, Ph.D.(Princeton) Asso. Professor, University of Maryland; Consultant, Naval Research Laboratory, Washington, D. C.

BERNARD VINOGRADE, Ph.D.(Michigan) Asso. Professor, Iowa State College, Ames, Iowa

LAURA B. WEAR, M.A.(Minnesota) Student, University of Pennsylvania, Philadelphia, Pa.

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R. C. WREDE, B.S.E.(Miami U.) Grad. Asst., Miami University, Oxford, Ohio.

MARTIN WRIGHT, M.A.(Texas) Asst. Professor, University of Houston, Tex.

W. W. YOUDEN, Student, Brown University, Providence, R. I.

LEO ZIPPIN, Ph.D.(Pennsylvania) Asso. Professor, Queens College, Flushing, L. I.

REPORT OF THE TREASURER FOR THE YEAR 1949

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1949. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1949

M & T Trust Co., Buffalo	$ 7,306.07	Current Fund	$ 7,306.07
Securities	61,885.56	Carus Fund	8,081.26
		Chace Fund	9,776.25
		Houck Fund	9,536.64
		Chauvenet Fund	624.58
		General Fund	33,866.83
	$69,191.63		$69,191.63

II. CURRENT FUND

Balance, January 1, 1949	$ 7,306.07	MONTHLY	
Dues	13,998.82	Publication	$11,999.63
Initiation fees	1,090.00	Reprints (net)	179.65
Subscriptions	3,978.16	Editor's office	511.34
Sale of back numbers (net)	1,291.56	Secretary-Treasurer's Office	
Advertisements	1,831.50	Clerical help	5,224.17
Contributions for publication of		Office expenses	2,111.78
the MONTHLY	400.00	Bank fee	59.70
Sales of exchange periodicals	55.50	Auditing fee	25.00
Interest on General Fund	1,143.71	Board of Governors	2,535.63
Income from Hardy Fund	120.00	Meetings	265.09
Transfer from General Fund	593.48	Representatives	45.73
		Subventions	925.00
		Balance, Dec. 31, 1949	7,926.08

III. CARUS FUND

Balance, January 1, 1949	$ 8,081.26	Reprinting 7th Monograph	$ 1,422.20
Sales of Monographs	3,296.94	Balance, January 31, 1949	10,337.74
Interest	270.34		
Increase in value of securities	111.40		

IV. CHACE FUND

Balance, January 1, 1949	$ 9,776.25	Honorarium, 2nd Slaught Paper	$ 114.00
Sale of Papyrus	300.00	Printing, 2nd Slaught Paper	2,156.68
Sale of Slaught Papers	727.76	Balance, Dec. 31, 1949	9,003.16
Interest	332.72		
Increase in value of securities	137.11		

V. HOUCK FUND

Balance, January 1, 1949	$ 9,536.64	Expenses of Index	$ 116.00
Interest	311.93	Balance, Dec. 31, 1949	9,861.11
Increase in value of securities	128.54		

VI. CHAUVENET FUND

Balance, January 1, 1949	$ 624.58	Balance, Dec. 31, 1949	$ 653.95
Interest	20.80		
Increase in value of securities	8.57		

VII. GENERAL FUND

Balance, January 1, 1949	$33,866.83	Transfer to Current Fund	$ 593.48
Increase in value of securities	471.32	Balance, Dec. 31, 1949	33,744.67

VIII. TOTAL FUNDS OF THE ASSOCIATION ON DEC. 31, 1949

Current Fund	$ 7,926.08	M & T Trust Company, Buffalo	
Carus Fund	10,337.74	Checking Account	$ 7,926.08
Chace Fund	9,003.16	Securities	63,600.63
Houck Fund	9,861.11		
Chauvenet Fund	653.95		
General Fund	33,744.67		
	$71,526.71		$71,526.71

THE MAY MEETING OF THE INDIANA SECTION and a CONFERENCE ON NONLINEAR PROBLEMS

The Spring meeting of the Indiana Section of the Mathematical Association of America was held May 7, 1949 at the University of Notre Dame, Notre Dame, Indiana, simultaneously with a Conference on Nonlinear Problems sponsored by the Department of Mathematics of the University of Notre Dame.

One hundred and one persons were present at the meeting, including the following forty-seven members of the Association: Rev. H. B. Albiser, Juna L. Beal, W. R. Beck, C. B. Bell, Stanley Bolks, Richard Brauer, H. E. Burns, G. E. Carscallen, H. E. Crull, Rev. H. F. DeBaggis, CSC, M. W. DeJonge, Sister M. Virgilia Dragowski, OSF, R. D. Edwards, Ky Fan, Cleota G. Fry, E. L. Godfrey, Michael Golomb, S. H. Gould, Noel Gottesman, G. R. Grainger, Sister M. Charlotte Holland, RSM, C. T. Hazard, Smith Higgins, Jr., Ralph Hull, H. F. S. Jonah, Wilfred Kaplan, M. W. Keller, E. L. Klinger, Rev. Bonaventure Knaebel, J. P. LaSalle, Florence Long, Sister M. Ferrer McFarland, RSM, P. M. Nastucoff, F. S. Nowlan, C. E. Olsen, R. R. Otter, P. W. Overman, P. M. Pepper, J. C. Polley, Arthur Rosenthal, A. E. Ross, G. X. Saltarelli, A. H. Smith, R. A. Struble, Anna K. Suter, W. D. Wood, L. J. Zimmerman.

Officers elected at the meeting were: Chairman, Ralph Hull, Purdue University; Vice-Chairman, J. C. Polley, Wabash College. P. M. Pepper, University of Notre Dame, continues as Secretary. The annual meeting in 1950 is to be held on Saturday, April 29, at Wabash College, Crawfordsville, Indiana.

Professor Richard Brauer, of the University of Michigan, gave an interesting and stimulating talk, entitled *Some Reflections on the Teaching of Algebra*, describing what he believes should be the aims in the teaching of this subject at the undergraduate level in the colleges. Professor Ralph Hull and Professor George Whaples officially led the animated discussion which followed Professor Brauer's provocative comments. The majority seemed to favor, with Professor Brauer, the placing of emphasis on understanding rather than on development of techniques with scant comprehension. Some persons were concerned that this emphasis might lead to the teaching of insufficient techniques. It was brought out in the subsequent remarks that the most satisfactory procedure would strive to accomplish both these aims, and that an attempt should be made to extend this attitude to the teaching in secondary schools.

Dr. Mary L. Cartwright, of Girton College, Cambridge, England, and

Princeton University, gave an address, *On Nonlinear Differential Equations and Some of Their Applications*. Dr. Cartwright illustrated a method for study of nearly linear resonance. This method (due to J. E. Littlewood and the speaker) was applied to the study of the equation $\ddot{x}-k(1-x^2)\dot{x}+x=pk\lambda\cos(\lambda t+\alpha)$, where k is small and λ is near 1. It was shown that many results can be obtained from the study of a related autonomous system.

Professor P. D. Edwards, Ball State Teachers College, Chairman of the *Committee on High School Mathematical Prerequisites for College Curricula in the State of Indiana*, gave a preliminary report of that committee. Responses have been received from all colleges in the state offering full four year degree courses, and from most others offering work of college grade. Variations existing in the requirements between colleges make the formation of general statements difficult. In most cases the only departments which do not require a knowledge of high school mathematics are English, Foreign Languages, Music, and Art. It is the hope of the Committee that the complete report may be made available to principals and student advisors in each high school in Indiana.

The following papers were presented:

1. *The eigenvalue problem for completely continuous normal operators*, by Professor Michael Golomb, Purdue University.

Most of the classical eigenvalue problems in differential and integral equations may be considered as special cases of the eigenvalue problem for completely continuous linear operators in Hilbert space. An operator is normal if it commutes with its adjoint. Important subclasses of the normal operators are the self-adjoint and the unitary operators. A simple proof is given for the theorem that every completely continuous normal operator has a complete system of eigenfunctions. Similar proofs have been known for completely continuous self-adjoint operators.

2. *The range of synchronization of subharmonic resonance*, by Mr. Hans Schaffner, Department of Electrical Engineering, University of Illinois, introduced by the Secretary.

Mr. Schaffner reported on the range of synchronization when $p\lambda=q\lambda_0$, where p and q are small integers, λ_0 is the resonant frequency, and λ is the frequency of the driving force.

3. *Numerical methods for solution of nonlinear differential equations*, by Professor Wilfred Kaplan, University of Michigan.

Professor Kaplan gave a general method for the analysis of differential equations $dx_i/dt=f_i(x_1, \cdots, x_n)$, $(i=1, \cdots, n)$ in a bounded phase space. The essential idea is to assume the f_i to be only approximately known. This leads to a blurring of the structure of the family of solutions; fine details are lost while only a simple structure remains; this is considered to be more in accordance with reality than the precise solutions of the exact equations. A numerical method for determining the stable states was given.

4. *Van der Pol's equation with forcing term*, by Dr. Mary L. Cartwright, Girton College, Cambridge, England, and Princeton University, introduced by Professor J. P. LaSalle.

It was suggested by van der Pol that for k very large the equation

$$\ddot{x}-k(1-x^2)\dot{x}+x=bk\lambda\cos(\lambda t+\alpha) \tag{1}$$

represents a physical system in which two stable subharmonics of period $(2n\pm1)2\pi/\lambda$ occur. This was investigated by J. E. Littlewood and the speaker. The stable solutions of (1) for k large and $0<b<2/3$ consist of descending waves from near $x=2$ to $x=1$, followed by a rapid descent to $x=-2$ approximately, and then a repetition with $-x$ for x. The waves from $x=2$ to $x=1$ are approximately solutions of $x-x^3/3=b \sin(\lambda t+\alpha)+C$, where C is a constant, and C is reduced by $k^{-1}\int_0^{2\pi/\lambda}x(t, C)dt$ after one complete period. There may be $n\pm(1/2)$ waves above $x=1$, and the same number below $x=-1$, where n is of the order of k, giving solutions with periods $(2n\pm1)2\pi/\lambda$. There are also unstable solutions with more complicated behavior near $x=\pm1$.

5. *Nonlinear circuit problems*, by Mr. Hans Schaffner, University of Illinois.

Mr. Schaffner outlined the nonlinear problems which appear important to the electrical engineers, and discussed their present status.

6. *Relaxation oscillations*, by Professor J. P. LaSalle, University of Notre Dame.

Professor LaSalle pointed out that the problem of proving the existence of periodic solutions of differential equations can be reduced in some cases to finding periodic solutions of suitable differential inequalities. The solutions of the inequalities provide a region of the phase plane which encloses a periodic solution (a closed path). This enclosure approximates the closed path and gives bounds for the amplitude and period of the periodic solution.

PAUL M. PEPPER, *Secretary*

CALENDAR OF FUTURE MEETINGS

International Congress of Mathematicians, Cambridge, Massachusetts, August 30–September 6, 1950.

Thirty-fourth Annual Meeting, University of Florida, Gainesville, December 30, 1950.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Allegheny College, Meadville, May 6, 1950

ILLINOIS, Southern Illinois University, Carbondale, May 12–13, 1950.

INDIANA, Wabash College, Crawfordsville, May 6, 1950.

IOWA, State University of Iowa, Iowa City, April 21–22, 1950.

KANSAS, State Teachers College, Pittsburg, April 22, 1950.

KENTUCKY, University of Kentucky, Lexington, April 29, 1950.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, University of Richmond, May 6, 1950.

METROPOLITAN NEW YORK, City College, April 1, 1950.

MICHIGAN

MINNESOTA, Macalester College, St. Paul, May 6, 1950.

MISSOURI

NEBRASKA, Nebraska Wesleyan University, Lincoln, May 6, 1950.

NORTHERN CALIFORNIA, University of San Francisco, January 27, 1951.

OHIO, Denison University, Granville, April 22, 1950.

OKLAHOMA

PACIFIC NORTHWEST, University of Washington, Seattle, June 16, 1950.

PHILADELPHIA, Lehigh University, Bethlehem, Pennsylvania, November 25, 1950.

ROCKY MOUNTAIN, University of Denver, April 28–29, 1950.

SOUTHEASTERN, University of Florida, Gainesville, April 7–8, 1950.

SOUTHERN CALIFORNIA

SOUTHWESTERN, Flagstaff, Arizona, April 29, 1950.

TEXAS, Abilene, April 14–15, 1950.

UPPER NEW YORK State, Syracuse University, April 22, 1950.

WISCONSIN, Marquette University, Milwaukee, May 13, 1950.

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